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SOLUTIONS OF LAPLACE'S EQUATION

One Dimension

Laplace equation $\nabla^2 V = 0$ reduces to an ordinary differential equation, if V is a function of a single variable.

In **cartesian coordinates** – If V is a function of x, then Laplace's equation reduces to

$$\partial^2 V/\partial x^2 = 0 \quad \text{or} \quad d^2V/dx^2 = 0 \qquad ...(1)$$

On integrating twice

$$V = Ax + B \qquad ...(2)$$

where A and B are constants to be determined from the boundary conditions.

In **cylindrical coordinates** – If V is a function of r, then Laplace's equation reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0. \qquad ...(3)$$

Multiplying by r and integrating,

$$r\frac{dV}{dr} = A \quad \text{or} \quad \frac{dV}{dr} = \frac{A}{r},$$

where A is a constant. On further integration,

$$V = A\log_e r + B. \qquad ...(4)$$

The constants A and B may be determined from the boundary conditions.

In **spherical coordinates** – If V is a function of r, then Laplace's equation reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0. \qquad ...(5)$$

Multiplying by r^2 and integrating,

$$r^2\frac{dV}{dr} = A \quad \text{or} \quad \frac{dV}{dr} = \frac{A}{r^2},$$

where A is a constant. On further integration,

$$V = -\frac{A}{r} + B. \qquad ...(6)$$

A and B are constants to be determined from the boundary conditions.

Two Dimensions

In **cartesian coordinates** – If V is a function of x and y, then Laplace's equation reduces to

$$\nabla^2 V = \partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0. \quad ...(7)$$

This equation can be solved by the method of separation of variables by assuming

$$V(x, y) = X(x)\,Y(y), \qquad ...(8)$$

where X is a function of x only and Y is a function of y only, substituting this value in equ -(7)

$$X''Y + Y''X = 0.$$

Dividing through by XY and separating X from Y, gives

$$-\frac{X''}{X} = \frac{Y''}{Y} = \lambda \text{ (suppose)}$$

where λ is a constant, known as separation constant.

$$X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0 \qquad ...(9)$$

There are following three possible cases depending on λ.

(a) $\lambda = 0$: If $\lambda = 0$, then from equation (9)

$$X'' = 0 \quad \text{or} \quad d^2X/dx^2 = 0$$

which on integration twice yields

$$X = Ax + B \quad \text{...(10)}$$

Similarly $$Y = Cy + D \quad \text{...(11)}$$

(*b*) $\lambda < 0$: If λ is negative, say $-\alpha^2$, then equation (9) becomes

$$X'' - \alpha^2 X = 0 \quad \Rightarrow \quad d^2X/\partial x^2 - \alpha^2 X = 0, \text{ or } (D^2 - \alpha^2)X = 0.$$

$$\therefore \quad DX = \pm\, \alpha X \quad \text{or} \quad dX/dx = \pm\, \alpha X.$$

Therefore there are two possible solutions corresponding to the plus and minus sign

$$\frac{dX}{dx} = \alpha X \quad \text{or} \quad \int \frac{dX}{X} = \alpha \int dx$$

$$\therefore \quad \log_e X = \alpha x + A = \alpha x + \log_e A_1$$

or $$X = A_1 e^{\alpha x}.$$

Similarly for the minus sign, the solution is given by

$$X = A_2 e^{-\alpha x}$$

$$\therefore \text{ Total solution } \quad X = A_1 e^{\alpha x} + A_2 e^{-\alpha x} \quad \text{...(12)}$$

$$= B_1 \cosh \alpha x + B_2 \sinh \alpha x. \quad \text{...(13)}$$

where $B_1 = A_1 + A_2$ and $B_2 = A_1 - A_2$.

(*c*) $\lambda > 0$: Let $\lambda = \beta^2$, then equation (9) becomes

$$X'' + \beta^2 X = 0 \quad \text{or} \quad (D^2 + \beta^2)X = 0$$

$$\therefore \quad DX = \pm\, j\, \beta X \quad \text{or} \quad dX/dx = \pm\, j\beta X. \quad \left(\because\ j = \sqrt{-1}\right)$$

There are two possible solutions. The total solution

$$X = C_1\, e^{j\beta x} + C_2\, e^{-j\beta x} \quad (14)$$

$$= D_1 \cos \beta x + D_2 \sin \beta x. \quad (15)$$

where $D_1 = C_1 + C_2$ and $D_2 = C_1 - jC_2$.

Therefore the general solution in two dimensional cartesian coordinates is

$$V = (B_1 \cosh \alpha x + B_2 \sinh \alpha x)\,(D_1 \cos \alpha y + D_2 \sin \alpha y) \quad \text{for } \alpha < 0$$

$$= (B_1' \cos \alpha x + B_2' \sin \alpha x)\,(D_1' \cosh \alpha y + D_2' \sinh \alpha y) \quad \text{for } \alpha > 0$$