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Program for B.Sc (Hons) Part-3

Theorms on coset

1. **If H is a subgroup of the group G, then**
   a. eH = He = H, where e is the identity element.
   b. hH = Hh =H, Where h ∈ H.

Proof : (i). We are given that 'e' is the identity element of G. therefore , we can write

eH = H and He = H. ⇒ eH = He = H. Proved

(ii) Here H is a subgroup of G. So H is closed with respect to the binary operation defined on G. Therefore the operation of elements of H by h will be the elements of H in same order.

So if a∈ G be any element and a ∈H also, then aH=H , Ha =H . This implies that aH=Ha=H

2. **If G is abelian, left coset of H in G = right cosets of H in G. Prove it by an example.**

Suppose G be the multiplicative group and G = {1, -1, i ,-i }. Also suppose that H = { 1, -1} be the subgroup of G. Now the left cosets of H in G are

1× H = { 1, -1} = H
-1 × H = { -1, 1} = {1, -1} = H
i × H = { i, -i}
-i × H = { -i, i} = { i, -i}

Here, we observe that 1xH and -1x H give the same set H since if h ∈ H , then h H = H.

Similarly i × H and -i × H give the same set. Thus there be formed two distinct left costs of H in G as {1, -1} and { i , i}.

Now, we can find the right cosets of H in G as follows:-

$H \times 1 = \{ 1, -1\} = H.$

$H \times (-1) = \{-1, 1\} = \{1, -1\} = H$

$H \times i = \{ i \times -i \}$

$H \times (-i) = \{ -i , i \} = \{ i , -i\}$

So, there are two right cosets of H in G are $H = \{ 1,-1\}$ and $\{ i , -i\}$.

This shows that the left cosets and right cosets coincides (i.e. equal), because the group G is Abelian.

Theorem 3. If a, b $\in$ G and $a \neq b$ then $aH=bH$ implies $b^{-1}a \in H$ and conversely.

Proof :- Let $bH = aH$ $\Rightarrow b^{-1} bH = b^{-1}aH$

$\Rightarrow H = b^{-1}aH$; Since $b^{-1} b = e$ and $eH = H$.

Again, as we know that $hH = H$, where $h \in H$.

Therefore , $h = b^{-1}a \in H \Rightarrow b^{-1} a \in H$. First part proved.

Conversely, let $b^{-1} a = h$ and $h \in H$.

Then $a H = (bb^{-1})a H$, $bb^{-1}=e$ and $e(aH)=eH$

$= b (bb^{-1}) aH$; $bb^{-1}=e$ and $e(aH)= eH$

$= b (b^{-1}a) H$ $= bhH$; $\because b^{-1}a=h$

$=bH$; $\because hH= H$ if $h \in H$. $\Rightarrow aH = bH$

Hence the proof.

Theorem 4 : If a, b $\in$ G, then $Ha=Hb \Rightarrow ab^{-1} \in H$ and conversely

Proof: $\rightarrow$ Let $Ha= Hb$.

$\Rightarrow Ha b^{-1} = Hb b^{-1}$ $\Rightarrow H ab^{-1} = H (bb^{-1}) = He$;

$\because b b^{-1} = e$ and $He = H$. $= H$ So, $ab^{-1} \in H$;

Since if $h \in H$, then $H h= H$.

Conversely, Suppose $ab^{-1} = h$ and $h \in H$. That is $ab^{-1} \in H$.

Then $Ha = Hae= Hab^{-1}b$

$= H (ab^{-1}) b$

$= H h b$; $ab^{-1} = h$

$= Hb$; $Hh = H$

$\Rightarrow Ha = Hb.$

Hence the proof.