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Program for B.Sc (Hons) Part-3

Index of a subgroup

Index of a subgroup in a group :- Index of a subgroup in a group is the number of distinct left (or right) cosets of H in G.

oR, The number of left (or right) cosets of H in G is said to be the index of H in G and is denoted by (G:H)

Examples :- 1. Let G be the additive group of integers and H the subgroup of even integers.

Now we can write G = { …. ,3, -2, -1, 0, 1,2,3 …….}

And H = { ……. , -2n, ……., -6, -4, -2, 0, 2, 4, 6, …-, 2n, ……..}

Here G is abelian since (a+ b) = (b+ a) ∀ a, b ∈ G. So any right coset is equal to the left coset.

Now, we get right costs of H in G as follows :-

O∈G; H + O = { …., -6, -4, -2, 0, 2, 4,6 ……} = H

1 ∈G ; and H+1 = { ……, -5,-3, -1, 1,3 ,5, 7 ……}

2∈G ; H+2 = { …….., -4, -2, 0, 2,4,6,8, ………} =H+O

3∈G ; H+3 = { ……… , -3, -1, 1, 3, 5 …………} = H+1

Simililarly, the right coset (H+4) coincide with (H+2) ; (H+5) with (H+3) and so on .

Again, the right cost H + (-1) coincide with (H+1); H+ (-2) Coincide with (H+2) and so on. Hence (G: H) =2 , i.e. (ℤ, 2ℤ) =2

(Kernel of a homomorphism):- Suppose G and H be the two groups and we define a homomorphism f:G→H. Then the kernel of f denoted by Ker f is defined as

$$\text{Ker } f = \{ x \in G : f(x) = e' \}.$$

It should be noted that kernel of homomorphism is always non-empty because f is homomorphism and f(e)=e', so e∈ Ker f.

Ex: - Let f: $IR_o \rightarrow IR_o$ be a homomorphism, where IR_o is the set of non-Zero real numbers and IR_o is the multiplicative group. Also $f(x) = x^2$

By the definition of ker f = { $x \in R_o : f(x)=e$ }; e=e' , f: $IR_o \rightarrow IR_o$

Here, the identity element of the group $(R_o, x) = 1$.

So, $f(x) = x^2 \Rightarrow x^2=1 \Rightarrow x=\pm 1$. Therefore , ker f = {1,-1}.

Theorem :- Suppose f : G → G' be a homomorphism with kernel K. Also suppose a ∈G is a given element such that f (a)=a', a'∈ G'. Then the collection of all those elements of G which have the image a' in G' is the coset Ka of K in G.

Proof :- Suppose e and e' be the identity elements of the groups G and G' respectively. Suppose a ∈G and f(a) = a' ∈G.

Suppose $f^{-1}(a') = \{ x \in G : f(x) = a' \}$

Now we have to prove that $f^{-1}(a') = Ka$.

Suppose y ∈ coset Ka. Then y= Ka for some k∈K

We have , f (y)= f(ka)= f(k).f (a) ; since f is a homomorphism.

= e' f(a) ; since k∈ ker K ⇒ f(k) = e'

= f(a) =a'

So, $y \in f^{-1}(a')$.

$\Rightarrow y \in Ka \Rightarrow y \in f^{-1}(a')$

Therefore $Ka \subseteq f^{-1}(a')$ ------------- (1)

Now, we are going to prove that $f^{-1}(a') \subseteq Ka$.

Suppose $z \in f^{-1}(a')$ be any element. Then f (z) = a'.

Now , we can write

$f(za^{-1}) = f(z) f(a^{-1})$; f is a homomorphism

$= f(z) [f(a)]^{-1}$; since $f(a^{-1}) = [f(a)]^{-1}$

$= a' \cdot (a')^{-1}$

$= e^{-1}$

Therefore , $za^{-1} \in K \Rightarrow (za^{-1}) a \in Ka \Rightarrow z \in Ka$.

So, $f^{-1}(a') \subseteq Ka$ --------------------- (2)

From (1) and (2), We have

$f^{-1}(a') = Ka$.

Hence the proof.