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## Properties of Coset

**Theorem 1.** :- There exists one-one onto mapping from H onto aH.

Proof :- Let G be a group and $H \leq G$. Now , we have to prove that there exists a mapping $f : H \to aH$, is bijection i.e. one-one onto . For this , we do the followings :-

(i) f is onto :- $\because f(h) = ah$

Again , since every element $ah \in aH$ is the f-image of $h \in H$. therefore f is onto.

(ii) f is one-one :- Let $ah_i = ah_j$ ; $i \neq j$

$\Rightarrow h_i = h_j$ ; by left cancellation law. $\Rightarrow$ f is one-one

H is finite subgroup; the number of elements in each of its left coset is equal to the no. of elements in H. (i.e. equal to the order of H)

**Note** :- There exists a bijection from H onto Ha. We can prove it by similar process of above theorem.

**Theorem 2.** If H is a subgroup of G, there exists a one-one mapping between two right cosets of H in G.

**Proof: -** Let G be a group and $H \leq G$. New, let $a,b \in G$ be a any two arbitrary elements. Then Ha and Hb are the two right cosets of H in G generated by a and b respectively.

Now, let $f : Ha \to Hb$ and is defined by $f(ha) = hb \; \forall \; h \in H$.

Now, we prove the followings :-

(i) **f is one-one** :- Suppose $h_1, h_2 \in H$. This implies that $h_1a, h_2a \in Ha$.

By the definition of f, we can write

$f(h_1a) = h_1b$ and $f(h_2a) = h_2b$.

Now, let $f(h_1a) = f(h_2a) \Rightarrow h_1b = h_2b$

$\Rightarrow h_1 = h_2$ ; by right cancellation law $\Rightarrow h_1a = h_2a$

Therefore f is one-one.

(ii) **f is onto** :- Suppose $h_3 \in Hb$ be any arbitrary element. Then $h_3b \in Hb \Rightarrow h_3 \in H \Rightarrow h_3 \in Ha$.

Now, $f(h_3a) = h_3b$ ; by the definition of f.

So, $h_3b \in Hb \Rightarrow h_3a \in Ha$ such that $f(h_3a) = h_3b$. Therefore f is onto.

Here , we also observe that the cosets Ha and Hb have the same no of elements if they are finite.

**Theorem 3.** Let G be a group and $H \leq G$. Then there exists a one-one correspondence between any two left cosets of H in G.

**Proof** :- Here, we are given that G is a group and $H \leq G$. Now let $a, b \in G$ be any two arbitrary elements. Then aH and bH are the two left cosets of H in G generated by a and b resfectively. Now. Let $f : aH \rightarrow bH$ and is defined by $f(ah) = bh \ \forall \ h \in H$. Then, we prove the followings:-

**(i)** **f is one-one** :- Let $h_1, h_2 \in H$. This implies that $ah_1, ah_2 \in H$. This implies that $ah_1, ah_2 \in Ha$. By the definition of f, we can write $f(ah_1) = bh_1$ and $f(ah_2) = bh_2$

Now, let $f(ah_1) = f(ah_2) \quad \Rightarrow bh_1 = bh_2$

$\Rightarrow h_1 = h_2$ ; by left cancellation law

$\Rightarrow ah_1 = ah_2$

Therefore f is one-one.

**(ii)** **f is onto** :- Let $h_3 \in bH$ be any arbitrary element . Then $bh_3 \in bH \Rightarrow h_3 \in H \Rightarrow h_3 a \in Ha$.

Now, $f(ah_3) = bh_3$ ; by the definition of f. So, $bh_3 \in bH \Rightarrow ah_3 \in aH$ such that $f(ah_3) = bh_3$. Therefore f is onto.

**Theorem 4.** Suppose G be a group and also suppose H be a subgroup of G. Then the no. of left cosets of H in G is equal to the no. of right cosets of H in G.

**Proof :-** Here, it is clear that G is a group and $H \leq G$. Now, we have to prove that the no. of left cosets of H in G is the same as the no. of right cosets of H in G. For this, we define a mapping f from the set of left costs of H in G to the set of right costs of H in G by the formula $f(aH) = Ha^{-1} \ \forall \ a \in G$.

**(i)** **f is one-one** :- Let $f(aH) = f(bH) \quad \Rightarrow Ha^{-1} = Hb^{-1}$ ; by the $def^n$ of f.

$\Rightarrow a^{-1}(b^{-1})^{-1} \in H$ ; since $Ha = Hb \Rightarrow ab^{-1} \in H$. $\quad \Rightarrow a^{-1}b \in H$

$\Rightarrow (a^{-1}b)H = H$ ; $\because \ h \in H \Rightarrow hH = H$. $\quad \Rightarrow a(a^{-1}b)H = aH$

$\Rightarrow bH = aH$, so f is one-one.

**(ii)** **f is onto** :- Let Ha be any right coset. Then $a^{-1}H$ is a left coset. Also $f(a^{-1}H) = H(a^{-1})^{-1}$, $def^n$ of f = Ha

So, every right coset Ha is the f-image of the left coset $a^{-1}H$. so f is onto. Now, it can be concluded that if the no fo distinct right costs in G is finite, then it is the same as the distinct left cosets of H in G.

**Theorem 5**:- If $a,b \in G$ and H is a subgroup of G, then

(i) $a \in bH \Rightarrow aH = bH$ and conversely.

(ii) $a \in bH \Rightarrow Ha = Hb$ and conversely.

**Proof of (i)** Here, we are given that G is a group and $H \leq G$. Also $a,b \in G$.

Now, let $a \in bH$. Then, we have to prove that $aH = bH$.

For this , Since $a \in bH$ $\Rightarrow b^{-1}a \in b^{-1}bH$

$\Rightarrow b^{-1}a \in eH$ ; $b^{-1}b = e$

$\Rightarrow (b^{-1}a)H = H$ ; $eH = H$

$\Rightarrow b(b^{-1}a)H = bH$

$\Rightarrow (bb^{-1})aH = bH$

$\Rightarrow eaH = bH$

$\Rightarrow aH = bH$

Conversely , let $aH = bH$ Since $a \in aH$

$\Rightarrow a \in bH$ ; Since $aH = bH$ Hence the proof

**Proof of (ii)** Let $a \in Hb$.

$\Rightarrow ab^{-1} \in Hbb^{-1} \Rightarrow ab^{-1} \in He$

$\Rightarrow ab^{-1} \in H$ $\Rightarrow H(ab^{-1}) = H$ ; since $h \in H \Rightarrow Hh = H$

$\Rightarrow H(ab^{-1})b = Hb$ $\Rightarrow Hae = Hb$ $\Rightarrow \boxed{Ha = Hb}$

Conversely, let $Ha = Hb$. Since $a \in Ha$

$\Rightarrow a \in Hb$ ; $\because Ha = Hb$ Hence the proof.

**Theorem 6** :- Any two left cosets of H in G are either disjoint or identical

**Proof** :- Suppose there be two left cosets aH and bH in G.

Again, suppose aH and bH are not disjoint .

Since aH and bH are not disjoint , therefore there exists an element c (say) such that $c \in aH$ and $c \in bH$.

Now, let $c = ah_1$ and $c = bh_2$ , where $h_1, h_2 \in H$.

$\Rightarrow ah_1 = bh_2$ ; $\because$ both are equal to c.

$\Rightarrow ah_1 h_1^{-1} = bh_2 h_1^{-1}$ $\Rightarrow ae = b(h_2 h_1^{-1})$ $\Rightarrow a = b(h_2 h_1^{-1})$

∵ H is a subgroup, therefore $h_2h_1^{-1} \in H$ because H is closed.

Suppose $h_2h_1^{-1} = h_3$. So $a = bh_3$

New, $aH = bh_3H = b\,(h_3H) = bH$ ; ∵ $h_3 \in H$ and $h_3H = H$

Therefore, we get two left cosets are indentical if they are not disjoint.

Hence , we conclude that either $aH \cap bH = \phi$ or $aH = bH$.

**Theorem 7**:- Any two right cosets of H in G are either disjoint or identical.

**Proof** :- Suppose there be two right cosets Ha and Hb in G. Again, Suppose that Ha and Hb are not disjoint.

Since Ha and Hb are not disjoint, therefore there exists and element c (say) such that $c \in Ha$ and $c \in Hb$.

Now, let $c = h_1a$ and $c = h_2b$, where

$h_1, h_2 \in H. \Rightarrow h_1a = h_2b$ ; ∵ both are equal to c

$\Rightarrow h_1^{-1}\,ha = h_1^{-1}h_2b$ $\Rightarrow ea = (h_1^{-1}h_2)b$ $\Rightarrow a = (h_1^{-1}h_2)b$

Since H is a subgroup of G, therefore $h_1^{-1}h_2 \in H$ because H is closed.

Let $h_1^{-1}h_2 = h_3$. So $a = h_3b$

Now, $Ha = Hh_3b$ $\Rightarrow Ha = Hb$ ; Since $h_3 \in H$ and $Hh_3 = H$.

So, we get an observation that two right cosets are indentical if there are not disjoint.

Hence we conclude that either $Ha \cap Hb = \phi$

or $Ha = Hb$.

Hence the proof.