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Program for B.Sc (Hons) Part-3

Cosets of Sub Group

Subgroup :- Let (G,∗) be a group and H a subset of G. Now H is called a subgroup of (G,∗) if, H is a group under the group operation ∗ of G. We may write it as H≤G.

From the above definition it is clear that the group G is itself a subgroup. Again, the identity group consisting by the identity element only, is the smallest subgroup. Generally G and {e} are called trivial subgroups. All other subgroup are called proper subgroup

Ex:- 1. The set of integer's ℤ is an additive group. Again, the set of even integers is a subset of ℤ and (E,+) is a group. So, (E,+) is a subgroup of (ℤ,+).

Ex:- 2. The set of real numbers IR is a subgroup of additive group of complex numbers ℂ.

Ex:- 3. The set R^* of non-zero real numbers is a group under multiplication. Again the set Q^* of non-zero rational number is a subset of R^* and is a group under multiplication Therefore Q^* is a subgroup of R^*

Necessary and sufficient Condition for a subgroup

Theorem 1. Let G be a group and H a non empty subset of G. Then H is a subgroup of G if, and only if

(i) ∀ a, b ∈ H ⇒ ab ∈ H (i.e. H is closed under ∗) and

(ii) ∀ a ∈ H ⇒ a^{-1} ∈ H. (i.e. existence of inverse element)

Theorem 2. Let G be a group and H a non-empty subset of G if and only if

$a, b \in H \Rightarrow ab^{-1} \in H$.

Theorem 3. A non empty subset H of finite group G is a subgroup of G if.

$a, b \in H \Rightarrow a\,b \in H$. Here H is closed under multiplication.

Theorem 4. If H_1 and H_2 be two subgroups of G, then $H_1 \cap H_2$ is also a subgroup of G.

Product of Two subgroups

Definition :- Let H and K be two subgroup of G. We define $HK=\{ h\,k : h \in H, K \in K\}$, then HK is a non-empty subset of G.

Theorem :-If H,K are two subgroups of G, then HK defined by $HK= \{ h\,k, h \in H: k \in K\}$ is a subgroup of G if HK=KH.

Cosets

Let H be subgroup of a group G and $a \in G$. Then the set $aH= \{ ah:h \in H\}$ is called a left coset of H in G.

Here aH means $a * h$ where $*$ is the binary operation of G.

Similarly $Ha = \{ ha: h \in H\}$ is called right cosets of H is G.

- If G is an additive group then left cosets aH is written a (a+H) and $a+H = \{ a+ h :h \in H\}$. Similarly Right cosets $H+a= \{h +a :h \in H\}$
- If G is an abelian group then every left coset is the same as the corresponding right coset . We simply called it cosets.

  Ex :- Consider the subgroup $H=3\mathbb{Z} = \{3k: K \in \mathbb{Z}\}$ of additive group $\mathbb{Z}$ then

$0+H = \{0 + 3k : K \in \mathbb{Z}\} = \{3k : K \in \mathbb{Z}\}$

$1+H = \{1 + 3k : K \in \mathbb{Z}\} = \{ \dots\dots\dots , -5, -2, 1, 4, 7 \dots\dots\dots\dots\dots \}$

$2+H = \{2 + 3k : K \in \mathbb{Z}\} = \{ \dots\dots , -4, -1, 2, 5, 8, \dots\dots\dots\dots\}$

$3+H = \{3 + 3k : K \in \mathbb{Z}\} = \{ \dots\dots , -6, -3, 0, 3, 6, 9 \dots\dots\dots \} = 0 + H$

$4+H = \{4 + 3k : K \in \mathbb{Z}\} = \{ \dots\dots\dots , -5, -2, 1, 4, 7, \dots\dots\dots\dots \} = 1 + H$

$5+H = \{5 + 3k : K \in \mathbb{Z}\} = \{ \dots\dots\dots , -1, 2, 5, 8, \dots\dots\dots\dots\dots \} = 2 + H$

Therefore, there are only three distinct left cosets of $3\mathbb{Z}$ in $\mathbb{Z}$ i.e. $0+H$, $1+H$, $2+H$.

Ex :- $H = 4\mathbb{Z} = \{ 4K : K \in \mathbb{Z}\}$.

Left cosets of $4\mathbb{Z}$,

$0+H = \{0 + 4K : K \in \mathbb{Z}\} = \{ 4K : K \in \mathbb{Z}\} = H$

$1+H = \{1+4K : K \in \mathbb{Z}\} = \{ \ldots\ldots , -3, 1, 5, 9 \ldots\ldots\}$

$2+H = \{2+4K : K \in \mathbb{Z}\} = \{ \ldots\ldots , -2, 2, 6, 10 \ldots\ldots\}$

$3+H = \{3+4K : K \in \mathbb{Z}\} = \{ \ldots\ldots , -1, 3, 7, 11 \ldots\ldots\}$

$4+H = \{4+4K : K \in \mathbb{Z}\} = \{ \ldots\ldots -4, 0, 4, 8, 12 \ldots\ldots\} = 0+H$

$5+H = \{5+4K : K \in \mathbb{Z}\} = \{ \ldots\ldots -3, 1, 5, 9, 13, \ldots\ldots\} = 1+H$

$6+H = \{6+4K : K \in \mathbb{Z}\} = \{ \ldots\ldots -2, 2, 6, 10, 14, \ldots\ldots\} = 2+H$

$7+H = \{7+4K : K \in \mathbb{Z}\} = \{ \ldots\ldots , -1, 3, 7, 11, 15 \ldots\ldots\} = 3+H$

There are only 4 left costs of $4\mathbb{Z}$ are $0+H$, $1+H$, $2+H$, $3+H$ (distinct)