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Program for B.Sc (Hons) Part-3

Coset Decomposition

Relation :- R ⊆ (A × B) or R ⊆ (B×A)

Ex :- A = { 1,2} , B {3}

(A×B) = { (1,3) , (2,3)· R= {(1,3) ⊆ (A×B)

Relation is also called Binary relation.

Relation on a set :- If A=B, then R ⊆ (A×A) or, R ⊆ (BXB). Here R is called relation on a set . Ex :- A { 1,2} ⇒ A×A = {(1,1) , (1,2) , (2,1) ,(2,2)}

Equivalence relation :- A relation R defined on A is called equivalence relation if the following axiom are satisfied :-

(i) R is reflexive :- For all a∈ A (a,a) ∈R. That is aRa ⊬ a∈A.

(ii) R is symmetric :- If (a,b)∈R⇒ (b,a)∈R where a, b ∈ A i.e. if aRb ⊬ a ∈ A.

(iii) R is transitive :- If (a,b)∈R, (b,c)∈R ⇒(a,c) ∈R, i.e aRb, bRc ⇒ aRc ; where a,b,c ∈ (a,c)∈ R.

Ex :- A = { 1,2,3}

R = { (1,1) , (2,2) (3,3), (1,2), (2,1)}

Equivalence class :- Let R ⊆ (A×A). That is R is a relation defined on A and R is also equivalence relation.

Now, let a∈A be an arbitrary element , then the set of all those elements of A, say (x) such that x R a, is called an equivalence class of A determined by a. Symbolically.

[a] = { x : x Ra} = {x ∈ A :(x, a)∈R}

Slly [b] = { x: x Rb} = { x∈ A : (x, b) ∈R}

Ex :- A = {a,b,c} and R = {(a,a), (b,b) , (c,c) (a,b) (b,a)}

Here R is an equivalence relation on A.

$[a] = \{ x: (x,a) \in R\} = \{a,b\}$

$[b] = \{x : (x,b) \in R\} = \{ b,a \} = \{a,b\}$

$[c]= \{ x: (x,c) \in R\} =\{c\}$

** Equivalence classes are either identical or disjoint. Also the union of equivalence classes determines the set on which R is defined.

Partition of a set :- A partition of a set A is a disjoint class C of non-empty subsets of A whose union gives the set A itself. The subsets are called the partition sets.

(i) $A_i \neq \not\subset$, $i \in IN$

(ii) $A_i \neq A_j$, $i \neq j$

(iii) $\underset{i \in IN}{U A_i} = A$

Ex :- $A= \{ a,b,c,d,e\}$

$A_1 = \{ a,b,d\}$, $A_2 = \{c\}$, $A_3 = \{e\}$

(A_1, A_2, A_3) is a partition of A.

Coset Decomposition :- Let G be group and $H\leq G$. Then for every element $a \in G$, there exists a left coset aH or right coset Ha respectively.

The set of all these left coset or right coset provides decomposition of G into disjoint equivalence classes such that for any two elements a,b of the some class (in the form of coset) $b^{-1}a$ (or ab^{-1}) $\in H$.

Theorem on coset Decomposition

Statement :- If H is a subgroup of the finite group G, then G is equal to the union of all left (right) cosets of H in G i.e. there exists a finite number of disjoint cosets of H in G say $a_1 H$, $a_2 H$, a_3H_2 ...... $a_k H$ in such a way that $G= a_1H \cup a_2H \cup a_3H \cup .... \cup a_kH$.

Proof :- We prove this theorem by the help of equivalence relation defined on G. Let G be a group and $H\leq G$. If $a, b \in G$, then we say that a is congruent to b modulo H iff $b^{-1}a \in H$. We can write $a\equiv b(\text{mod } H)$ if $b^{-1} a \in H$ i.e. $a=bh$; $(h\in H)$.

Now, we have to slow that this congruence relation is an equivalence relation on G. For this, we do the followings:-

(i) Reflexive :- It is clear that $a \equiv a \pmod{H}$; since $a = ae$, $e \in H$.

(ii) Symmetric :- Let $a \equiv b \pmod{H}$. Now we show that $b \equiv a \pmod{H}$

$\because a \equiv b \pmod{H}$

$\Rightarrow b^{-1} a \in H \Rightarrow (b^{-1}a)^{-1} \in H$; Since H is a group

$\Rightarrow a^{-1}b \in H$

$\Rightarrow b \equiv a \pmod{H}$

(iii) **Transitive** :- If $a \equiv b \pmod{H}$ and $b \equiv c \pmod{H}$, then we can write $a = bh_1$, and $b = ch_2$ where $h_1, h_2 \in H$ Therefore $a = (ch_2)h_1 = c(h_2 h_1) = ch_3$; Since h_3 $h_3 \in H$ and put $h_2h_1 = h_3$

$\Rightarrow a = c\, h_3$

$\Rightarrow a \equiv c \pmod{H}$.

Hence , we observe that congruence relation in G is an equivalence relation . The equivalence relation partitions the group G into disjoint classes which are left cosets of H in G. So, by the properties of equivalence classes, we get $G = a_1 H \cup a_2 H \cup a_3 H$ .... a_kH; K = no. of distinct left cosets of G

We may prove the relation $ab^{-1} \in H$ is also an equivalence relation in G for sight costs also.