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Program for B.Sc (Hons) Part-3

## Topic :- Elementary Properties of Automorphism :-

**Property 1 :-** If T ϵ Aut (G) , then

i. T(e) = e

ii. O (T(a)) = o(a), where a ϵG is of order o(a)>o.

**Proof :-** (i) As we know that if f: G →H is a homomorphism, then

f (e)= $\bar{e}$; $\bar{e}$ ϵ H.

Here T : G→G is homomorphism, so T (e)= e , since $\bar{e}$ = e in G.

(ii) Let o (a) = n so that n is the least positive integer such that

$a^n = e$ -------------> (1)

Now $a^n = e$ ⇒ T ($a^n$) = T (e)

⇒ T (a.a.a………….. n times) = T (e)

⇒ T (a) T (a) T (a)………….. n times) = T (e),

Since T is a homomorphism

⇒ $\{T(a)\}^n = e$ -------------> (2)

Now, we prove that η is the least positive inlayer such that a satisfying equation (2).

If possible, let $\{T(a)\}^m = e$ for some positive integer a m, o<m<n.

Then T (a) T(a) – m times =e

⇒ T (a. a. a …….. m times ) =e, since T is homomorphism.

⇒ T ($a^m$) = e = T(e) ; from (1)

⇒ $a^m = e$ , o<m<n, since T is one- one. This makes a contradiction for (1).

Therefore o(T(a)) = n= o (a).

**Property 2**: Prove that a group G is Abelign if and only if a→$a^{-1}$ is an automorphism.

Solution :- Suppose f: G→G be such that f (x)=$x^{-1}$ for all x ϵ G.

Suppose that G is abelian.

Now, we prove that f is an automorphism.

The function $f$ is one - one since

$f(x) = f(y) \Rightarrow x^{-1}=y^{-1} \Rightarrow (x^{-1})^{-1} = (y^{-1})^{-1} \Rightarrow x=y.$

Also, if $x\epsilon$G (second group ), then $x^{-1}\epsilon$G and

we have $f(x^{-1})^{-1} = (x^{-1})^{-1} = x$.

Therefore $f$ is onto.

Thus $f$ is one- one and onto.

Now , suppose G is abelian. Consider a,bϵG.

Then $f$ (ab) = $(ab)^{-1}$ ; by definition of $f$.

$= b^{-1}a^{-1} = a^{-1}b^{-1}$ ; *G is abelion*

$= f(a) f(b)$

∴ $f$ is an automorphism of G.

Conversely, suppose that $f$ is an automorphism of G. we shall prove that G is ablian.

We have , $f(ab) = (ab)^{-1}$ ; by the definction of $f$.

$= b^{-1} a^{-1}$

$= f(b) f(a)$ ; by definition of $f$.

$= f(ba)$ ;

∵ $f$ is an automorphism

Since $f$ is one-one , therefore.

$f(ab) = f(ba) \Rightarrow ab = ba \Rightarrow G$ is abelian.

**Property 3**. Let G be a finite abelian group of order $n$ . Now, the mapping $\sigma : x \rightarrow x^m \forall x \in G$.

(i) Let x ,y ϵ G.

Then $\sigma(xy) = (xy)^m = x^m y^m$, since G is abelian

$= \sigma(x)\,\sigma(y)$

∴ σ is homomorphism.

(ii) **σ is onto**:

Since $(m,n) = 1$, there exist integers $r$ and $s$ such that $mr + ns = 1$.

Let $x \in G$. then $x = x^1 = x^{mr+ns} = x^{mr}.\, x^{ns}\, (x^r)^m\, (x^n)^s$,

*Where* $x^n = e$

$\therefore\ x = x^{mr} \forall x \in G$ ------------------------- *(1)*

$\Rightarrow x = (x^r)^m \ \forall x \in G$

$\Rightarrow x = (y)^m$ , *where* $y = x^r \in G$

$\therefore x = \sigma(y)$ , $y \in G$ .

Hence σ is onto.

(iii) **σ is one – one** : To prove σ is one – one, we prove that ker (σ) = {e}

Let g ϵ ker (σ) be arbitrary. Then σ (g) = e.

$\Rightarrow \sigma^m = e \Rightarrow g^{mr} = e^r = e \Rightarrow g = e$, by (1)

⇒ ker (g) = {e} = σ is one-one .

Hence σ is an automorphism of G.

**Property 4**: If a be any fixed element of a group G then the mapping $T_a : G \rightarrow G$ defined by

$T_a(x) = axa^{-1} \forall x \in G$ is an automorphism of G.

Proof :- The given mapping is

$T_a : G \rightarrow G$ defined by

$T_a(x) = axa^{-1} \forall x \in G$. Now, we prove $T_a$ is an automorphism .

For this, we prove the followings:-

(i) $T_a$ is a homomorphism :-

Let $x,y$ be any two elements of G . Then

$T_a(xy) = a(xy)a^{-1}$ ,by definition

$= a(xa^{-1}ay)a^{-1} = (axa^{-1})(aya^{-1})$ ; $\because a^{-1}a = e$

$= T_a(x)\, T_a(y)$

$\therefore$ $T_a$ is a homomorphism.

(ii) $T_a$ is one - one : we have,

$T_a(x) = T_a(y) \Rightarrow axa^{-1} = aya^{-1}$

$\Rightarrow x = y$ ; by cancellation law in $G$

$\therefore$ $T_a$ is 1-1.

(iii) **$T_a$ is onto** : Let $g \in G$ be arbitrary. Then

$g = a(a^{-1}ga)\, a^{-1}$ ; Since $aa^{-1} = e$

$= ag_1a^{-1}$ , where $g_1 = a^{-1}\, ga \in G$

$\Rightarrow g = T_a(g_1)$ where $g_1 \in G$

$\therefore$ $T_a$ is onto.

Therefore $T_a$ is an automorphism of G.