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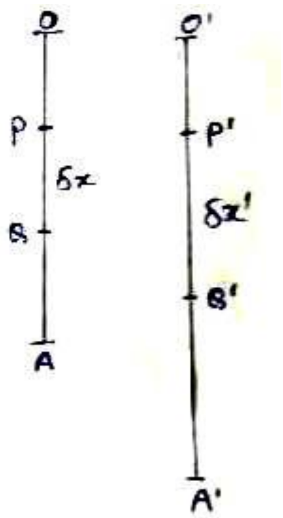
Program for B.Sc (Hons) Part-2

Topic: - Numericals On Hooke's Law

Ex : 1. Prove that the modulus of an elastic string is equal to the force which would stretch a light string to twice its natural length.



Sol^n :- Suppose the natural length of the string be l. Now suppose a weight W be applied to the free end of the string. Suppose that an element PQ (=δx) is stretched to a length P'Q'= $\delta x'$ The tension in the string Q'P' is given by $\lambda.\left(\frac{\delta x' - \delta x}{\delta x}\right)$. For equilibrium of the element

$\delta x'$, This tension is equal and opposite to the weight W attached to A'. So

$$\lambda.\left(\frac{\delta x' - \delta x}{\delta x}\right) = W \Rightarrow \frac{\delta x'}{\delta x} - \frac{\delta x}{\delta x} = \frac{W}{\lambda}$$

$$\Rightarrow \frac{\delta x'}{\delta x} = 1 + \frac{W}{\lambda} \Rightarrow \delta x^1 \left(1 + \frac{W}{\lambda}\right) \delta x.$$

For the extension of the length l to $2l$,

$$\int_o^{2l} dx' = \int_o^{l} \left(\frac{W}{\lambda} + 1\right) dx \Rightarrow 2l = \left(\frac{W}{\lambda} l + l\right)$$

$$\Rightarrow 2l - l = \frac{W}{\lambda}.l \Rightarrow l = \frac{W}{\lambda}.l \Rightarrow 1 = \frac{W}{\lambda} \Rightarrow \lambda = W.$$

That is the modulus of elasticity is equal to the stretching force W.

Ex :-2 Find the work done in extending an elastic light string to double its natural length.

Sol^h :- Suppose the natural length of the given string be l. Again , if the string be extended to a length x,

Then the tension T is given by

$$T = \lambda\left(\frac{x - l}{l}\right); \text{ by Hooke's Law.}$$

The work done against the tension T in stretching to a further length δx is

$$T.\,\delta x = \lambda\left(\frac{x-l}{l}\right).\,\delta x\,.$$

Therefore the work done in extension from a length l to a length 2l

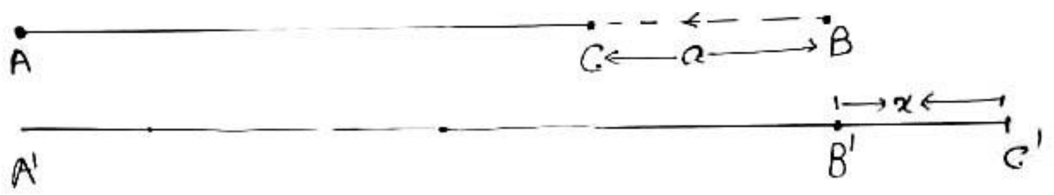
$$= \int_l^{2l} \frac{\lambda}{l}\,(x-l)dx = \frac{\lambda}{l}\left\{\left[\frac{x^2}{2}\right]_l^{2l} - [lx]_l^{2l}\right\}$$

$$= \frac{\lambda}{l}\left[\frac{4l^2}{2} - \frac{l^2}{2} - 2l^2 + l^2\right]$$

$$= \frac{\lambda}{l}\cdot\frac{l^2}{2} = \frac{\lambda l}{2}$$

Ex. 3 A light string is kept compressed by the action of a given force. When the force is suddenly reversed, Proof that the greatest subsequent extension of the spring is three times its initial contradiction.

Sul[h] :- Suppose l be the natural length of the spring and F be the give force. Also , let BC (=a) be the length through which the spring is kept compressed by F. Then for equilibrium , F = force of the spring against compression,



According to Hooke's Law,

$$F = \lambda\,.\frac{a}{l}\,;$$ where λ is the modulus of compression.

Now, the force F is suddenly reversed, then let the extension be B'C' $= x$, *(say)*

Here, we consider the motion of an element of mass m at C' in the time t. The force of extension is $\lambda\cdot\frac{a}{l}$ and this is acting against extension is $\lambda\cdot\frac{x}{l}$.

So, the equation of motion is $m\frac{d^2x}{dt^2} = \frac{\lambda a}{l} - \frac{\lambda x}{l}$

$$\Rightarrow\ m\,V\,\frac{dv}{dx} = \frac{-\lambda}{l}\,(x-a)$$

$$\Rightarrow\ m\,V.\,dv = \frac{-\lambda}{l}\,(x-a)dx.$$

On integration, we get $\frac{1}{2}$ $mV^2 = \frac{-\lambda}{2l}(x-a)^2 + \frac{C}{2}$,

Where c is a constant.

When x = -a =BC, then V =o, as the element 'm' is at rest.

So $c = \frac{4a^2\lambda}{l}$

$$\Rightarrow c = mv^2 = \frac{4a^2\lambda}{l} - \frac{\lambda}{l}(x-a)^2$$

But putting V = o, we get the required greatest

extension ,

$(x-a)^2 = 4a^2 \Rightarrow (x-a) = \pm 2a \Rightarrow x = 3a$ *or,*

$x = -a$

Rejecting the negative value of x, as the extension can not be negative, we get

$x = 3a$

This shows that the greatest extension of the spring is three times its initial contradiction.

Ex :-4 . A uniform flexible chain of length ℓ rest in a straight line on a smooth horizontal table except for a length a which hangs over an edge at right angles to it. The chain moves from rest, show that after a time t the length x of the over - hanging portion given by $x = a\cosh\left(\sqrt{\frac{g}{l}}\,.t\right)$

Solution: - The equation of motion can be written as

$$l.\frac{d^2x}{dt^2} = gx$$

$$\Rightarrow 2\frac{dx}{dt}.l.\frac{d^2x}{dt^2} = 2\frac{dx}{dt}.gx$$

$$\Rightarrow l.\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = g.\frac{d}{dt}(x^2).$$

On integration , we get $l\left(\frac{dx}{dt}\right)^2 = gx^2 + C$; where C is the content of integration

Now if $\frac{dx}{dt} = o, x = a, \Rightarrow C = -gx^2$

So, $\quad l.\frac{d^2x}{dt^2} = g(x^2 - a^2)$

$$\Rightarrow \frac{dx}{dt} = \sqrt{\frac{g}{l}}.\sqrt{x^2 - a^2} \Rightarrow \sqrt{\frac{g}{l}}.dt = \frac{dx}{\sqrt{x^2-a^2}}$$

on integration , we get

$$\sqrt{\frac{g}{l}}.\mathrm{t} = \cos\ h^{-1}\left(\frac{x}{a}\right) + k \text{ ; where k is constant.}$$

For the initial case, *t=0*, *x=a* ⇒ k =0

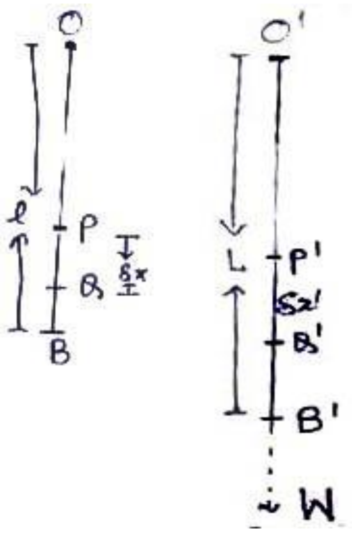
$$\sqrt{\frac{g}{l}}.\mathrm{t} = \cos\ h^{-1}\left(\frac{x}{a}\right)$$

Therefore $\quad x = a \cos h\left(\sqrt{\frac{g}{l}}\ .\mathrm{t}\right).$

Ex : 5 (i) Prove that the extension of heavy elastic string of weight w and natural length l hanging from one end and supporting a weight W at the other is $\frac{l}{\lambda}\left(\mathrm{W} + \frac{1}{2}\mathrm{w}\right)$, where λ is the modulus of elasticity of the string.

(ii) Find the extension in a heavy elastic string due to its own weight when the strings hangs vertically.

(iii) Prove that the extension of a heavy elastic string hanging vertically from one end, due to its own weight is the same as if the string were light and a weight equal to half its weight were supported at its lower extremity.



Solve : Suppose OB = unstretched string and OB' =stretched string (of length L)

Again, suppose the element PQ of length δ becomes P'Q' of length δx' when stretched .

For equilibrium ,

$$\lambda.\frac{\delta x' - \delta \mathrm{x}}{\delta \mathrm{x}} = W + \frac{w}{l}\,(l - x)$$

$$\Rightarrow \quad \delta x' = \left[1 + \frac{\mathrm{W}}{\lambda} + \frac{\mathrm{w}}{l\lambda}\,(l - x)\right]\delta x$$

$$\Rightarrow \quad \int_{\mathrm{o}}^{\mathrm{L}} \mathrm{d}x' = \int_{o}^{l}\left[1 + \frac{W}{\lambda} + \frac{w}{l\lambda}\,(l - x)\right]dx$$

$$\Rightarrow \quad \mathrm{L} = l + \frac{\mathrm{W}}{\lambda}l + \frac{\mathrm{w}}{\lambda}l - \frac{-\mathrm{w}}{2\lambda}l$$

$$\Rightarrow \quad \mathrm{L} = l + \frac{l}{\lambda}\left(\mathrm{W} + \frac{\mathrm{w}}{2}\right).$$

Therefore the required extension $= \frac{l}{\lambda}\left(W + \frac{w}{2}\right) \rightarrow$ **(1)**

(ii) Here , W=o. So the extension becomes

$$\frac{l}{\lambda}\left(o + \frac{w}{2}\right) = \frac{l}{2\lambda}w. \rightarrow \textbf{(2)}$$

(iii) In the first case W =o, so the extension becomes $\frac{l}{2\lambda}w$.

From the result (1) of part – ( ).

In the second case w =o and W $= \frac{1}{2}$w,

then the extension $= \frac{l}{2\lambda}W \rightarrow$ **(3)**

clearly (2) and (3) are the same. Hence the result.

Ex :6 An elastic string of natural length 2L can support a certain weight when it is stretched till its length is 3L. One end of the string is how attached to a fixed point on a smooth horizontal table and same weight is attached to the other end. Prove that if the weight be pulled to any distance and let go. The string will become slack after a time $\frac{\pi}{2}\sqrt{\frac{L}{g}}$.

Solution : We are given that, the natural length of string = 2L,

Stretched length of the string = 3L. So, the extension becomes (3L-2L) =L

By Hooke's Law, tension T =mg $= \frac{\lambda L}{2L} \Rightarrow \lambda = 2mg$,

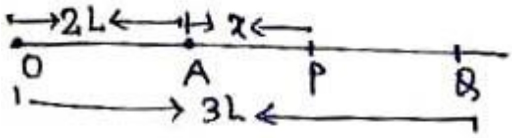
Where λ = modulus of elasticity, m = vertically hanging mass.

Suppose the string be attached to O and mass m is attached to A and pulled up to Q and then released. Also suppose P be the position at time t such that AP $=x$. So, the equation of motion can be written as

$$\frac{d^2x}{dt^2} = \frac{\lambda x}{2L} \Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{g}{L}\right)x \,;\, \lambda = 2mg$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\mu x \,; where\ \mu = \frac{g}{L}.$$

This equation shows that motion is S.H.M about the centre at A. The string becomes slack when particle crosses A.



$\therefore$ The required time, T$= \frac{1}{4}\cdot\frac{2\pi}{\sqrt{\mu}} = \frac{\pi}{2}\sqrt{\frac{L}{g}}$.