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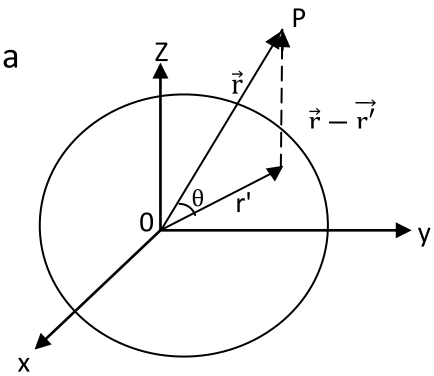
Multipole expansion of potential due to charge distribution :-



Consider a small element of volume dv' at a distance $\vec{r'}$ from origin in a continuous charge distibution.

Let P be a point at a distance $\vec{r}$ from origin

ρ =Volume charge density

Charge of volume element dv' =dq

$$dq = \rho dv'$$

Potential at P due to charge in elementary volume

$$d\phi = \frac{1}{4\pi\varepsilon_0} \frac{dq}{|\vec{r}-\vec{r}'|}$$

$$d\phi = \frac{1}{4\pi\varepsilon_0} \frac{\rho dv'}{|\vec{r}-\vec{r}'|}$$

Potential at P due to whole charge distribution

$$\phi = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho dv'}{\left|\vec{r}-\vec{r}'\right|} \quad \text{----------------------------} \; (1)$$

But $|\vec{r}-\vec{r}'|^2 = (\vec{r}-\vec{r}').(\vec{r}-\vec{r}')$

$$= r^2 + r'^2 - 2\,\vec{r}\cdot\vec{r'}$$

$$= r^2 + r'^2 - 2r\cdot r' \cos\theta$$

$$|\vec{r}-\vec{r}'| = (r^2 + r'^2 - 2\,rr'\,\cos\theta)^{\frac{1}{2}}$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = (r^2 + r'^2 - 2\,rr'\,\cos\theta)^{\frac{-1}{2}}$$

Putting this value in eqn. (1)

$$\phi=\frac{1}{4\pi\varepsilon_0}\int_V \rho dv'\,(r^2+r'^2-2rr'\cos\theta)^{\frac{-1}{2}}$$

$$\phi=\frac{1}{4\pi\varepsilon_0}\int_V \rho dv'\left[r^2\left(1+\frac{r'^2}{r^2}-\frac{2r'\cos\theta}{r}\right)\right]^{\frac{-1}{2}}$$

$$\phi=\frac{1}{4\pi\varepsilon_0 r}\int_V \rho dv'\left[1+\left(\frac{r'^2}{r^2}-\frac{2r'}{r}\cos\theta\right)\right]^{\frac{-1}{2}} \text{ -------------- (2)}$$

$$\because \quad \left[1+\left(\frac{r'^2}{r^2}-\frac{2r'}{r}\cos\theta\right)\right]^{\frac{-1}{2}}=1-\frac{1}{2}\frac{r'^2}{r^2}+\frac{r'}{r}\cos\theta+\frac{3}{8}\left(\frac{2r'}{r}\right)^2\cos^2\theta+--$$

$$=1+\frac{r'}{r}\cos\theta+\left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2-1}{2}\right)+----$$

Putting this value in eqn. (2)

$$\phi=\frac{1}{4\pi\varepsilon_0 r}\int_V \rho dv'\left(1+\frac{r'}{r}\cos\theta+\left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\theta-1}{2}\right)\right)+\text{------}$$

$$\phi=\frac{1}{4\pi\varepsilon_0 r}\int_V \rho dv'+\frac{1}{4\pi\varepsilon_0 r^2}\int_V r'\cos\theta\,\rho dv'+\frac{1}{4\pi\varepsilon_0 r^3}\int_V r'^2\left(\frac{3\cos^2\theta-1}{2}\right)\rho dv'+---(3)$$

$$\phi=\frac{K_1}{4\pi\varepsilon_0 r}+\frac{K_2}{4\pi\varepsilon_0 r^2}+\frac{K_3}{4\pi\varepsilon_0 r^3}+$$

where K_1=Monopole moment, K_2 = Dipole moment

K_3 = Quadrupole moment

$$\therefore \quad \phi=\phi_1+\phi_2+\phi_3 \quad \text{------------------ (4)}$$

$$\because \phi_1=\frac{1}{4\pi\varepsilon_0}\frac{K_1}{r}=\frac{1}{4\pi\varepsilon_0 r}\int_V \rho dv'$$

$$\because \int_V \rho dv'=\int_V dq=q=\text{Total charge distribution}$$

$$\phi_1=\frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad \text{-----------------} \quad (5)$$

This potential is known as monopole potential

$$\because \phi_2=\frac{1}{4\pi\varepsilon_0}\frac{K_2}{r}=\frac{1}{4\pi\varepsilon_0 r^2}\int_V r'\cos\theta\,\rho dv'$$

$$= \frac{1}{4\pi\varepsilon_0 r^2} \int_V \frac{\vec{r}.\vec{r}'}{r} \rho dv'$$

$$\phi_2 = = \frac{1}{4\pi\varepsilon_0 r^2} \int_V \frac{\vec{r}.\vec{r}'}{r} \rho dv' \text{ --------------- } (6)$$

The potential at any point of position vector $\vec{r}$ due to dipole of dipole moment $\vec{p}$ is given by $= \frac{1}{4\pi\varepsilon_0} \frac{\vec{r}.\vec{P}}{r^3}$ ------------------ (7)

Comparing eqn. (6) & (7)

$$\int_V \vec{r}' \rho dv' = \vec{P} \text{ is known as the dipole moment of charge distribution}$$

$$\phi_2 = \frac{1}{4\pi\varepsilon_0} \frac{\vec{r}.\vec{P}}{r^3}$$

$$\phi_2 = \frac{1}{4\pi\varepsilon_0} \frac{\hat{r}.\vec{P}}{r^2}$$

This potential is known as dipole potential

$$\because \phi_3 = \frac{1}{4\pi\varepsilon_0} \frac{K_3}{r^3} = \frac{1}{4\pi\varepsilon_0 r^3} \int_V r'^2 \rho dv' \left(\frac{3\cos^2\theta - 1}{2}\right)$$

The quantity within the integrals in known as quadrupole moment. The potential due to qurdrupole moment varies as $\frac{1}{r^3}$.

This is known as quadrupole potential.

Thus eqn. (4) follows that the potential at any point due to arbitrary charge distribution is the sum of the potentials due to monopole, dipole, quardrupole and so on moments all assumed to be placed at origin.