Dr. ASHOK KUMAR

Guest faulty, Department of Physics

Magadh Mahila Collage, P.U.

Name of Program –Physics (Hons), Part III

Paper –VII, Group A

Magnetohydrodynamic equations

The motion of the normal fluid is described by the equations of hydrodynamics which represent the conservation laws of mechanics. The conservation of matter is described by the law of continuity. If ρ is the density and v the velocity

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho v) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \text{ div } v + v \text{ grad } \rho = 0. \quad \ldots(1)$$

The second equation of hydrodynamics relates to gain of momentum in a fluid element to the forces which act upon it. Hence

$$\rho\left(\frac{dv}{dt}\right) = - \text{grad } P + \rho g + F \quad \ldots(2)$$

where P is the pressure, F is the viscous force per unit volume and the gravitational force is represented by ρg where g is the acceleration due to gravity, $\frac{d}{dt}$ is the mobile operator so that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \text{ grad.}$$

Consider the electromagnetic effect

Magnetic stress $(\frac{i}{c} \times B)$ This means that we have to utilize the generalised version of Ohm's law in which the electric field is taken as that seen by an observer moving with the fluid. Hence the forces on a unit charge moving with the fluid with velocity v is the force as due to sum of the electric field and Lorentz field. As it is unit charge the total electric field will be $\left(E + \frac{v}{c} \times B\right)$. Then

$$E^* = E + \frac{v}{c} \times B \quad \ldots(3)$$

$$i = \sigma\left[E + \frac{v}{c} \times B\right] \quad \ldots(4)$$

If the fluid is a perfect conductor $\frac{i}{\sigma} = 0$ Hence

$$E + \frac{v}{c} \times B = 0 \quad \ldots(5)$$

Neglecting the gravitational and viscous forces in equation... (2) include the force due to electromagnetic effect

$$\rho\left(\frac{dv}{dt}\right) = - \text{grad } P + \frac{1}{c}(i \times B)$$

or

$$\rho\left[\frac{\partial v}{\partial t} + v \text{ grad } v\right] = - \text{grad } P + \frac{1}{c}(i \times B) \quad \ldots(6)$$

Maxwell's equations are

$$\frac{1}{\mu} \text{curl } B = \frac{4\pi i}{c} \quad \ldots(7)$$

$$\text{curl } E = -\frac{1}{c} \cdot \frac{dB}{dt} \quad \ldots(8)$$

$$\text{div } B = 0 \quad \ldots(9)$$

$$\text{div } E = \frac{4\pi e}{\varepsilon} \quad \ldots(10)$$

From equation (1) if ρ is invariant

$$\rho \text{ div } v = 0$$

or
$$\text{div } v = 0$$

Since a conducting fluid moving in a magnetic field.

Then transform the equations in terms of the parameters, ρ, v, P and B and eliminate E and i from equation (3.7)

$$i = \frac{c}{\mu}\frac{1}{4\pi} \text{ curl } B$$

Putting the value of i in equation (6)

$$\rho\left[\frac{\partial v}{\partial t} + v \text{ grad } v\right] = -\text{ grad } P + \frac{1}{c}\left[\frac{c}{\mu}\frac{1}{4\pi}\text{curl} B \times B\right]$$

$$= -\text{ grad } P + \frac{1}{4\pi\mu}[\text{curl} B \times B]$$

$$= -\nabla P - \frac{1}{8\pi\mu}\cdot \nabla B^2 + \frac{1}{4\pi\mu} B\cdot\nabla B \qquad \text{...(11)}$$

From equation (7)

$$\frac{1}{\mu}\text{curl}(\text{curl} B) = \frac{4\pi}{c}\text{ curl } i$$

$$= \frac{4\pi}{c}\text{ curl}\left[\sigma\left\{E + \frac{v}{c}\times B\right\}\right]$$

$$\frac{1}{\mu}[\text{grad div} B - \nabla^2 B] = \frac{4\pi\sigma}{c}\left[\text{curl} E + \text{curl}\frac{v\times B}{c}\right]$$

as div $B = 0$ equation (9)

$$-\frac{1}{\mu}\nabla^2 B = \frac{4\pi\sigma}{c^2}[c \text{ curl} E + \text{curl}(v\times B)]$$

$$\frac{1}{\mu}\nabla^2 B = \frac{4\pi\sigma}{c^2}\left[\frac{\partial B}{\partial t} - \text{curl}(v\times B)\right]$$

as curl $(v \times B) = v$ div $B - B$ div $v - (v \text{ grad}) B + (B \text{ grad}) v$

then
$$\frac{1}{\mu}\nabla^2 B = \frac{4\pi\sigma}{c^2}\left[\frac{\partial B}{\partial t} - v \text{ div} B + B \text{ div } v + (v\nabla) B - (B\nabla) v\right]$$

Since div $B = 0$

$$\frac{1}{\mu}\nabla^2 B = \frac{4\pi\sigma}{c^2}\left[\frac{\partial B}{\partial t} + B \text{ div } v + (v\nabla) B - (B\nabla) v\right]$$

$$\frac{c^2}{4\pi\sigma\mu}\nabla^2 B = \frac{\partial B}{\partial t} + B \text{ div } v + (v\nabla) B - (B\nabla) v \qquad \text{... (12)}$$

Thus the modified MHD equations as follows :

$$\frac{\partial\rho}{\partial t} + \rho \text{ div } v + v \text{ grad } \rho = 0$$

$$\rho\left[\frac{\partial v}{\partial t} + v\nabla v\right] = -\nabla P - \frac{1}{8\pi\mu}\cdot\nabla B^2 + \frac{1}{4\pi\mu}B\cdot\nabla B$$

$$\frac{\partial B}{\partial t} + B \text{ div } v + (v\nabla) B - (B\nabla) v = \frac{c^2}{4\pi\sigma\mu}\nabla^2 B$$

The RHS becomes zero when $\sigma \to \infty$

$$\text{div } B = 0$$

$$\text{div } v = 0$$