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Program for B.Sc (Hons) Part-3

Topic: - Inner Automorphism

Definition :- Suppose a be an element of a group G Now, the automorphism $T_a: G \longrightarrow G$ defined by $T_a(x) = a\,x\,a^{-t}$ $\forall\, x \in G$ is called on inner automorphism of G induced by a.

The set of all automorphisms defined on G is denoted by I (G). That is I (G) is the collection of all automorphisms of G. Moreover $I(G) = \{T_a : a \in G\}$.

The inner automorphism T_a is actually an automorphism of G. We can prove it as follows :-

$T_a(x) = a\,x\,a^{-1}$ $\forall\, x \in G$.

(i) T_a is operation preserving :-

Let $x_1, x_2 \in G$ be any two elements . Then

$$T_a(x_1 x_2) = a(x_1 x_2)a^{-1}$$

$$= a(x_1 a^{-1} a x_2)a^{-1};$$

$$= (a\,x_1\,a^{-1})(a\,x_2\,a^{-1}) \qquad \because a^{-1}a=e$$

$$= T_a(x_1)\,T_a(x_2)$$

Thus T_a is operation preserving.

(ii) T_a is one-one :-

Let $x_1\,x_2 \in G$ and $T_a(x_1) = T_a(x_2)$.

$\Rightarrow a\,x_1\,a^{-1} = a x_2\,a^{-1}$

$\Rightarrow x_1 = x_2$; by the law of cancellation in G.

Thus T_a is one . one mapping.

(iii) T_a is onto :-

Let $g \in G$ be an arbitrary element. Then

$g = a (a^{-1} ga) a^{-1}$; $\because aa^{-1} = e$

$= ag_1a^{-1}$; Put $g_1 = a^{-1}ga \in G$.

$\Rightarrow g = T_a (g_1)$ where $g_1 \in G$.

Thus T_a is onto.

Hence T_a is an automorphism of G.

Examples of inner automorphism :-

Ex :- The action of the inner automorphism of D_4 induced by R_{90} is given below.

$$x \xrightarrow{\emptyset R_{100}} R_{qo}\, x\, R_{qo}^{-1}$$

$R_{9o} \rightarrow R_{9o}\, R_{o}\, R_{90}^{-1} = R_o$

$R_{9o} \rightarrow R_{9o}\, R_{90}\, R_{90}^{-1} = R_{90}$

$R_{180} \rightarrow R_{9o}\, R_{180}\, R_{90}^{-1} = R_{180}$

$R_{270} \rightarrow R_{9o}\, HR_{90}^{-1} = R_{270}$

$H \rightarrow R_{9o}\, H_{270}\, R_{90}^{-1} = V$

$V \rightarrow R_{9o}\, V R_{90}^{-1} = H$

$D \rightarrow R_{9o}\, D R_{90}^{-1} = D^1$

$D^1 \rightarrow R_{9o}\, D^1 R_{90}^{-1} = D$

Ex :-2 Aut $(S_3) = S_3$.

As we know that

$S_3 \{ e, (1\ 2), (2\ 3), (3\ 1), (1\ 2\ 3), (1\ 3\ 2)\}$

Also, we know inner automorphism. According to inner automorphism , if $a \in G$ be *a* fixed element, then the function $T_a : G \rightarrow G$ defined by

$T_a (x) = axa^{-1} \not\forall\, x \in G$ is an automorphism of G

Now, we take (i) a = e. Then $T_e(x) = x \ \forall x \in G$,

(ii) a = (1 2) . Then for $x \in G$,

We have $T_a(x) = (1\ 2)\ e\ (12)^{-1} = e$

$= (1\ 2)\ (2\ 3)\ (1\ 2)^{-1} = (2\ 3)$

$= (1\ 2)\ (3\ 1)\ (1\ 2)^{-1} = (3\ 1)$

$= (1\ 2)\ (1\ 2\ 3)\ (1\ 2)^{-1} = (1\ 2\ 3)$

$= (1\ 2)\ (1\ 3\ 2)\ (1\ 2)^{-1} = (1\ 3\ 2)$

Thus $T_{(12)}(x) = \{ e, (1\ 2), (2\ 3), (3\ 1), (1\ 2\ 3), (1\ 3\ 2)\}$

..... $T_{(2\ 3)}(x) = \{ e, (1\ 2), (2\ 3), (3\ 1), (1\ 2\ 3), (1\ 3\ 2)\}$

$T_{(31)}(x) = \{ e, (1\ 2), (2\ 3), (3\ 1), (1\ 2\ 3), (1\ 3\ 2)\}$

and so on.

Therefore Aut $(S_3) = \{ e, (1\ 2), (2\ 3), (3\ 1), (1\ 2\ 3), (1\ 3\ 2)\} = S_3$

Theorem on Inner automosphism

Statement :- Let G be a group. Then the collection of all the inner automorphisms of G (Say I (G)) is a normal subgroup of Aut (G).

Proof :- We prove the theorem in the two steps

:- (i) first we show that I (G) is a subgroup of Aut (G)

And (ii) I (G) is a normal subgroup of G.

(i) Let $T_a, T_b \in I(G)$.

Now for $x \in G$, we get $T_aT_b(x) = T_a(T_b(x))$

$= T_a(bxb^{-1})$

$= a\ (bxb^{-1})\ a^{-1}$; by def^n of inner automorphism

$= (ab)\ x\ (b^{-1}a^{-1})$

$= (ab)\ x\ (ab)^{-1}$; for a group $(ab)^{-1} = b^{-1}a^{-1}$

$= T_{ab}(x)$

$\therefore T_a\ T_b(x) = Tab\ (x) \ \forall\ a, b \in G$ ------------> (1)

From (1) $T_a T_b (x) \in I(G)$. Also $T_a T_{a^{-1}} = T_{aa^{-1}} = T_e$,

Where $T_e(x) = exe^{-1} = x = I(x)$, $x \in G$

$\Rightarrow T_e = I$ and so $T_a T_{a^{-1}} = I$ ----------> (2)

Form (2), it is clear that $(T_a)^{-1} \in I(G) \nvdash T_a \in I(G)$ --------> (3)

Hence from (2) and (3)

I (G) is a subgroup of Aut G.

(ii) I (G) is a normal subgroup of G -->

Let $x \in G$ be an arbitrary element. Then

$TT_aT^{-1}(x) = TT(T^{-1}(x))$

$= TT_a(y)$, Where $y = T^{-1}(x) \in G$

$= T\{T_a(y)\} = T(aya^{-1})$ by, definition

$= T(a)\,T(y)\,T(a^{-1})$, since T is a homomorphism

$= T(a)\,T\{T^{-1}(x)\}\{T(a)\}^{-1}$; by putting the value y.

$= b\{TT^{-1}(x)\}b^{-1}$, where $b = T(a) \in G$

$= bI(x)\,b^{-1}$, Since $TT^{-1} = I$

$= b\,x\,b^{-1} = T_b(x)$.

$\therefore TT_aT^{-1} = T_b \in I(G)$; Since $b \in G$.

This shows that I (G) is a normal subgroup of Aut (G).