Different Forms of Least Action Principle:
B.Sc. Part-3, Hons.

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Different Forms of Least Action Principle

The principle of least action can be expressed in different forms. If the transformation equations do not depend on time explicitly, then the kinetic energy is a quadratic function of the generalized velocities. In such a case from we have

$$\sum_i p_i \dot{q}_i = \sum_i \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i = 2T \qquad (1)$$

Equation least acti now reduces to

$$\Delta \int_{t_1}^{t_2} T\, dt = 0 \qquad (2)$$

This is another form of the principle of least action. Also if there is no external force on the system, T and H are conserved and the principle of least action takes the form

$$\Delta \int_{t_1}^{t_2} dt = 0 \quad \text{or} \quad \Delta(t_2 - t_1) = 0 \qquad (3)$$

$$t_2 - t_1 = \text{an extremum}$$

That is, of all paths possible between two points that are consistent with the conservation of energy, the system moves along the path for which the time of transit is the least. In this form, the principle is similar to **Fermat's principle** in geometrical optics, which states that a light ray travels between two points along such a path that the time taken is the least.

Again, when the transformation equations do not involve time, the kinetic energy is given by:

$$T = \sum_j \sum_k a_{jk}\, \dot{q}_j\, \dot{q}_k \qquad (4)$$

A configuration space for which the a_{ij} coefficients form the metric tensor can be constructed. The element of path length $d\rho$ in this space is defined by

$$(d\rho)^2 = \sum_j \sum_k a_{jk}\, dq_j\, dq_k \qquad (5)$$

$$\left(\frac{d\rho}{dt}\right)^2 = \sum_j \sum_k a_{jk}\, \dot{q}_j\, \dot{q}_k \qquad (6)$$

From Eqs. (4) and (6)

$$T = \left(\frac{d\rho}{dt}\right)^2 \quad \text{or} \quad dt = \frac{d\rho}{\sqrt{T}} \qquad (6a)$$

Equation (6a) helps us to change the variable in (2) and the principle of least action takes the form

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$$\Delta \int_{t_1}^{t_2} T dt = \Delta \int_{t_1}^{t_2} \sqrt{T}\; d\rho = 0 \qquad (7)$$

For conservative systems, $H = T + V$. Consequently, Eq. (7) becomes

$$\Delta \int_{t_1}^{t_2} \sqrt{H - V(q)}\; d\rho = 0 \qquad (8)$$

Eq 8 is referred to as **Jacobi's form of least action principle**.