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Program for B.Sc (Hons) Part-1

Topic : An Introduction to Lattice

Supremum and Infimum of a set :- Let A b a non-empty subset of X. Also, let (X,≤) be a partially ordered set.

If there exists an element b ∈X such that every x∈A is less than or equal to b i.e. x≤b, then b is called the least upper bound or supremum of and is denoted by sup (A) .

Similarly if there exists an element a ∈X such that a≤ x ⩲ x ∈ A, then a is called greatest lower bound of A and is denoted by inf. (A). Infimum of A is the synonyms of greatest lower bound.

Finally, we come on a conclusion that supremum is the smallest upper bound among so many upper bunds and infinum is the greatest among lower bounds. It can also be observed supremum and infimum may or may not belong to the concerned set.

Ex :- 1 Let |R = set of real numbers and let A⊆ |R such that

$A = \{x \in |R : 2 < x < 3\}$.

Now A^* = collection of all upper bounds

$= \{x \in |R : x \geq 3\}$

A_* = Collection of all lower bounds

$= \{ x\, x \in |R : x \leq 2\}$

So, inf. (A)= 2 and sup (A)=3. It is also noticed that neither 2 nor 3 ∈ A. But it does not mean that infinum or supemum never belongs to A.

There may chance that inf (A) or sup (A) or both ∈ A.

Definition of lattice :- A partially ordered set (L,≤) is called a lattice if each pair set { x,y} of elements L has a greatest lower bound as well as a least upper bound.

If and y are any two elements of L, the g. l. b and l .u. b of the set { x,y} are denoted by $x \wedge y$ and $x \vee y$ respectively.

Thus, we can say that a lattice is an abstract structure consisting by a partially ordered set in which every two elements have a unique supremum (also called least upper bound or join) and a unique infimum (also called a greatest lower bound or meet).

Examples on Lattice

E:- 1. Suppose (|R, ≤) be the partial ordered set, where ≤ is the used order relation on |R.

Here for any x, y ∈ |R , we can see that

$x \wedge y = \min(x, y)$

$x \vee y = \max(x, y)$

By the definition of lattice, it is clear that (|R , ≤) is a lattice.

Ex :- 2. Suppose { P (x), ⊆} be a partial ordered set, where X be a non-empty set and P (X) is the power set of X.

Here for any A, B∈ P (X), we can see that

$A \wedge B = A \cap B \in P(X)$

$A \vee B = A \cup B \in P(X)$

So , by the definition of lattice, { P (X), ⊆} is a lattice.

Ex :- 3 Suppose (|N, ≤) be a partial ordered set, where m, n∈ |N , m ≤n means that m divides n.

Here for any m, n ∈ |N,

$m \wedge n$= greatest common divisor of m and n

$m \vee n$= least common multiple of m and n

Also $(m \wedge n) \in$ |N and $(m \vee n) \in$ |N . Therefore (|N; ≤) is a lattice.

Example of a partially order set which is not a lattice

Ex :- 4 Suppose X= { 1,2,3,4,5,6,7,8} be a set.

Now, for $m, n \in$ X, let $m \leq n$ mean that m divides .

It is very clear that (X, $\leq$) is a partially ordered set.

But 4v6 = least common multiple of 4 and 6

= 12, which does not belong to X.

This proves that (X, $\leq$) is not a lattice even (X, $\leq$) is a partially ordered set. The reason behind it is that l. u .b for 4 and 6 $\notin$ X.

Theorem on (X, $\leq$).

Statement :- For any partially ordered set (X, $\leq$) ; the following conditions are equivalent :-

(i) Every non-empty subset of X which has an upper bound has a supremum.

(ii) Every non-empty subset of X which has a lower bound has an infimum.

Proof : It can be proved easily .