Problems on complex Variable function

(B.A/B.Sc, Part-III, Hons.)

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Example 4. *Determine whether $\frac{1}{z}$ is analytic or not ?*

Solution. Let $w = f(z) = u + iv = \frac{1}{z} \Rightarrow u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

Equating real and imaginary parts, we get

$$u = \frac{x}{x^2+y^2}, \qquad v = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \qquad \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus C – R equations are satisfied. Also partial derivatives are continuous except at (0, 0).

Therefore $\frac{1}{z}$ is analytic everywhere except at $z = 0$.

Also $$\frac{dw}{dz} = -\frac{1}{z^2}$$

This again shows that $\frac{dw}{dz}$ exists everywhere except at $z = 0$. Hence $\frac{1}{z}$ is analytic everywhere except at $z = 0$. **Ans.**

Example 5. *Show that the function e^x (cos y + i sin y) is an analytic function, find its derivative.*

Solution. Let $e^x(\cos y + i\sin y) = u + iv$

So, $e^x \cos y = u$ and $e^x \sin y = v$ then $\frac{\partial u}{\partial x} = e^x \cos y$, $\frac{\partial v}{\partial y} = e^x \cos y$

$$\frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

Here we see that $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are $C - R$ equations and are satisfied and the partial derivatives are continuous.

Hence, $e^x(\cos y + i\sin y)$ is analytic.

$$f(z) = u + iv = e^x(\cos y + i\sin y) \text{ and } \frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial v}{\partial x} = e^x \sin y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^x(\cos y + i\sin y) = e^x.e^{iy} = e^{x+iy} = e^z.$$

Which is the required derivative. **Ans.**

Example 6. *Test the analyticity of the function w = sin z and hence derive that:*

$$\frac{d}{dz}(\sin z) = \cos z$$

Solution. $w = \sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y, \quad v = \cos x \sinh y \quad \begin{bmatrix} \cos iy = \cosh y \\ \sin iy = i \sinh y \end{bmatrix}$

$\frac{\partial u}{\partial x} = \cos x \cosh y, \ \frac{\partial u}{\partial y} = \sin x \sinh y$

$\frac{\partial v}{\partial x} = -\sin x \sinh y, \quad \frac{\partial v}{\partial y} = \cos x \cosh y$

Thus $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad$ and $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

So $C - R$ equations are satisfied and partial derivatives are continuous.

Hence, $\sin z$ is an analytic function.

$$\frac{d}{dz}(\sin z) = \frac{d}{dz}[\sin x \cosh y + i \cos x \sinh y]$$

$$= \frac{\partial}{\partial x}(\sin x \cosh y + i \cos x \sinh y)$$

$$= \cos x \cosh y - i \sin x \sinh y = \cos x \cos iy - \sin x \sin iy$$

$$= \cos(x + iy) = \cos z$$

Ans.

$\cosh x = \frac{e^x + e^{-x}}{2} \quad \ldots (1)$

$\sinh x = \frac{e^x - e^{-x}}{2} \quad \ldots (2)$

$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \ldots (3)$

$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \ldots (4)$

From (1) $\cosh ix = \frac{e^{ix} + e^{-ix}}{2} = \cos x$

From (3) $\cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$

From (4) $\sin ix = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = i\,\frac{e^x - e^{-x}}{2} = i \sinh x$

From (2) $\sinh ix = \frac{e^{ix} - e^{-ix}}{2} = i \sin x$

Example 7. *Show that the real and imaginary parts of the function w = log z satisfy the Cauchy-Riemann equations when z is not zero. Find its derivative.*

Solution. To separate the real and imaginary parts of $\log z$, we put $x = r\cos\theta$; $y = r\sin\theta$

$$w = \log z = \log(x + iy)$$

$$\Rightarrow \quad u + iv = \log(r\cos\theta + ir\sin\theta) = \log r(\cos\theta + i\sin\theta) = \log_e r.e^{i\theta}$$

$$= \log_e r + \log_e e^{i\theta} = \log r + i\theta = \log\sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x} \qquad \begin{bmatrix} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\frac{y}{x} \end{bmatrix}$$

So $\quad u = \log\sqrt{x^2 + y^2} = \frac{1}{2}\log(x^2 + y^2), \ v = \tan^{-1}\frac{y}{x}$

On differentiating u, v, we get

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2 + y^2}.(2x) = \frac{x}{x^2 + y^2} \qquad \ldots (1)$$

$$\frac{\partial v}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2} \qquad \ldots (2)$$

From (1) and (2), $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \ldots (A)$

Again differentiating u, v, we have

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2 + y^2}(2y) = \frac{y}{x^2 + y^2} \qquad \ldots (3)$$

$$\frac{\partial v}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2} \qquad \ldots (4)$$

From (3) and (4), we have

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{... (B)}$$

Equations (A) and (B) are $C - R$ equations and partial derivatives are continuous. Hence, $w = \log z$ is an analytic function except

when $\quad x^2 + y^2 = 0 \Rightarrow x = y = 0 \Rightarrow x + iy = 0 \Rightarrow z = 0$

Now $\quad w = u + iv$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}$$

Which is the required derivative. **Ans.**

Example 8. *Find the values of* C_1 and C_2 *such that the function*

$f(z) = x^2 + C_1 y^2 - 2xy + i(C_2x^2 - y^2 + 2xy)$ *is analytic. Also find* $f'(z)$.

Solution. Let $\quad f(z) = u + iv = x^2 + C_1y^2 - 2xy + i(C_2x^2 - y^2 + 2xy)$

Equating real and imaginary parts, we get

$$u = x^2 + C_1y^2 - 2xy \text{ and } v = C_2x^2 - y^2 + 2xy$$

$$\frac{\partial u}{\partial x} = 2x - 2y \text{ and } \frac{\partial v}{\partial x} = 2C_2x + 2y$$

$$\frac{\partial u}{\partial y} = 2C_1y - 2x \text{ and } \frac{\partial v}{\partial y} = -2y + 2x$$

$C - R$ equations are

$$\left.\begin{array}{l}\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}\\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}\end{array}\right] \Rightarrow \begin{array}{ll}2x - 2y = -2y + 2x & \text{...(1)}\\ 2C_1y - 2x = -2C_2x - 2y & \text{...(2)}\end{array}$$

From (2) equating the coefficient of x and y.

$$2C_1 = -2 \quad \Rightarrow C_1 = -1$$

$$-2 = -2C_2 \Rightarrow C_2 = 1$$

Hence, $\quad C_1 = -1$ and $C_2 = 1$ **Ans.**

On putting the value of C_2, we get

$$\frac{\partial u}{\partial x} = 2x - 2y, \qquad \frac{\partial v}{\partial x} = 2x + 2y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (2x-2y) + i(2x+2y) = 2[(x+ix) + (-y+iy)]$$

$$= 2[(1+i)x + i(1+i)y] = 2(1+i)(x+iy) = 2(1+i)z$$

This is the required derivative. **Ans.**

Example 9. *Discuss the analyticity of the function* $f(z) = z\bar{z}$.

Solution. $\quad f(z) = z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2 + y^2$

$$f(z) = x^2 + y^2 = u + iv.$$

$$u = x^2 + y^2,\ v = 0$$

At origin, $$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h,\ 0) - u(0,\ 0)}{h} = \lim_{h \to 0} \frac{h^2}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0, 0+k) - u(0, 0)}{k} = \lim_{k \to 0} \frac{k^2}{k} = 0$$

Also, $$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h, 0) - v(0, 0)}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0, 0+k) - v(0, 0)}{k} = 0$$

Thus, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence, C – R equations are satisfied at the origin.

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{(x^2 + y^2) - 0}{x + iy}$$

Let $z \to 0$ along the line $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{(x^2 + m^2 x^2)}{(x + imx)} = \lim_{x \to 0} \frac{(1 + m^2)x}{1 + im} = 0$$

Therefore, $f'(0)$ is unique. Hence the function $f(z)$ is analytic at $z = 0$. **Ans.**

Example 10. *Show that the function $f(z) = u + iv$, where*

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, \quad z \neq 0$$
$$= 0, \qquad z = 0$$

satisfies the Cauchy-Riemann equations at $z = 0$. Is the function analytic at $z = 0$? Justify your answer.

Solution. $$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = u + iv$$

$$u = \frac{x^3 - y^3}{x^2 + y^2}, \qquad v = \frac{x^3 + y^3}{x^2 + y^2}$$

[By differentiation the value of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at (0, 0) we get $\frac{0}{0}$, so we apply first principle method]

At the origin

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h,\ 0) - u(0,\ 0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{h^2}}{h} = 1 \quad \text{(Along } x\text{- axis)}$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0,\ 0+k) - u(0,\ 0)}{k} = \lim_{k \to 0} \frac{\frac{-k^3}{k^2}}{k} = -1 \quad \text{(Along } y\text{- axis)}$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h,\ 0) - v(0,\ 0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{h^2}}{h} = 1 \quad \text{(Along } x\text{- axis)}$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,\ 0+k) - v(0,\ 0)}{k} = \lim_{k \to 0} \frac{\frac{k^3}{k^2}}{k} = 1 \quad \text{(Along } y\text{-axis)}$$

Thus we see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, Cauchy-Riemann equations are satisfied at $z = 0$.

Again
$$f'(0) = \lim_{z \to 0} \frac{f(0+z) - f(0)}{z} = \lim_{z \to 0} \left[\frac{\frac{x^3 - y^3 + i\,(x^3 + y^3)}{x^2 + y^2} - (0)}{x + i\,y} \right]$$

$$= \lim_{z \to 0} \left[\frac{x^3 - y^3 + i(x^3 + y^3)}{x^2 + y^2} \cdot \frac{1}{x + iy} \right]$$

Now let $z \to 0$ along $y = x$, then

$$f'(0) = \lim_{x \to 0} \frac{x^3 - x^3 + i(x^3 + x^3)}{x^2 + x^2} \left(\frac{1}{x + ix} \right)$$

$$= \frac{2i}{2(1+i)} = \frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{i+1}{1+1} = \frac{1}{2}(1+i) \qquad \ldots (1)$$

Again let $z \to 0$ along $y = 0$, then

$$f'(0) = \lim_{x \to 0} \frac{x^3 + ix^3}{x^2} \cdot \frac{1}{x} = (1+i) \qquad [\text{Increment} = z] \qquad \ldots (2)$$

From (1) and (2), we see that $f'(0)$ is not unique. Hence the function $f(z)$ is not analytic at $z = 0$. **Ans.**

Example 11. *Show that the function defined by* $f(z) = \sqrt{|xy|}$ *Satisfies Cauchy-Riemann equation at the origin but is not analytic at that point.*

Solution. Let $\quad f(z) = u + iv = \sqrt{|xy|}$

Equating real and imaginary parts, we get $u = \sqrt{|xy|}$, $v = 0$

At origin

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h,\, 0) - u(0,\, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0,\, 0+k) - u(0,\, 0)}{k} = \lim_{k \to 0} \frac{0 - 0}{k} = 0$$

Also
$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h,\, 0) - v(0,\, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,\, 0+k) - v(0,\, 0)}{k} = \lim_{k \to 0} \frac{0 - 0}{k} = 0$$

From the above results, it is clear that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, $C - R$ equations are satisfied at the origin

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

Let $z \to 0$ along the line $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{|mx^2| - 0}}{x(1+im)} = \lim_{x \to 0} \frac{\sqrt{|m|}}{1+im}$$

Thus, the limit on R.H.S. depends upon m and hence will have different values for different values of m.

Therefore, $f'(0)$ is not unique.

Hence the function $f(z)$ is not analytic at $z = 0$. **Ans.**

Example 12. *Show that the function*

$$f(z) = e^{-z^{-4}}, \quad (z \neq 0) \quad \text{and}$$
$$f(0) = 0$$

is not analytic at $z = 0$,

although, Cauchy-Riemann equations are satisfied at the point. How would you explain this.

Solution. $$f(z) = u + iv = e^{-z^{-4}} = e^{-(x+iy)^{-4}} = e^{-\frac{1}{(x+iy)^4}}$$

$$\Rightarrow \quad u + iv = e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}} = e^{-\frac{1}{(x^2+y^2)^4}[(x^4+y^4-6x^2y^2) - i4xy(x^2-y^2)]}$$

$$\Rightarrow \quad u + iv = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} . e^{\frac{-i4xy(x^2-y^2)}{(x^2+y^2)^4}}$$

$$\Rightarrow \quad u + iv = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\left[\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4} - i\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]}$$

Equating real and imaginary parts, we get

$$u = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}}, \quad v = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}}$$

At $z = 0$
$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h,\ 0) - u(0,\ 0)}{h} = \lim_{h \to 0} \frac{e^{-h^{-4}}}{h} = \lim_{h \to 0} \frac{1}{h e^{\frac{1}{h^4}}}$$

$$= \lim_{h \to 0} \left[\frac{1}{h\left[1 + \frac{1}{h^4} + \frac{1}{2!h^8} + \frac{1}{3!h^{12}} + \ldots\right]}\right], \quad \left(e^x = 1 + x + \frac{x^2}{2!} + \ldots\right)$$

$$= \lim_{h \to 0} \left[\frac{1}{\left[h + \frac{1}{h^3} + \frac{1}{2h^7} + \frac{1}{6h^{11}} \ldots\right]}\right] = \frac{1}{0 + \infty} = \frac{1}{\infty} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0,\ 0+k) - u(0,\ 0)}{k} = \lim_{k \to 0} \frac{e^{-k^{-4}}}{k} = \lim_{k \to 0} \frac{1}{k\ e^{\frac{1}{k^4}}} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h,\ 0) - v(0,\ 0)}{h} = \lim_{h \to 0} \frac{e^{-h^{-4}}}{h} = \lim_{h \to 0} \frac{1}{h.e^{\frac{1}{h^4}}} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0, 0+k) - v(0, 0)}{k} = \lim_{k \to 0} \frac{e^{-k^{-4}}}{k} = \lim_{k \to 0} \frac{1}{k \cdot e^{\frac{1}{k^4}}} = 0$$

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ($C - R$ equations are satisfied at $z = 0$)

But $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{e^{-z^{-4}}}{z}$$

Along $z = re^{i\frac{\pi}{4}}$ $$f'(0) = \lim_{r \to 0} \frac{e^{-r^{-4}} \cdot e^{-\left(e^{i\frac{\pi}{4}}\right)^{-4}}}{re^{i\frac{\pi}{4}}} = \lim_{r \to 0} \frac{e^{-r^{-4}} \cdot e^{-\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{-4}}}{re^{i\frac{\pi}{4}}}$$

$$= \lim_{r \to 0} \frac{e^{-r^{-4}} e^{-\cos\pi}}{re^{i\frac{\pi}{4}}} = \lim_{r \to 0} \frac{e^{-r^{-4}} \cdot e}{re^{i\frac{\pi}{4}}} = \infty$$

Showing that $f'(z)$ does not exist at $z = 0$. Hence $f(z)$ is not analytic at $z = 0$. **Proved.**

Example 13. *Examine the nature of the function*

$$f(z) = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}} ; z \neq 0$$
$$f(0) = 0$$

in the region including the origin.

Solution. Here $$f(z) = u + iv = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}} ; z \neq 0$$

Equating real and imaginary parts, we get

$$u = \frac{x^3 y^5}{x^4 + y^{10}}, \quad v = \frac{x^2 y^6}{x^4 + y^{10}}$$

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0 + h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^4}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0, 0 + k) - u(0, 0)}{k} = \lim_{k \to 0} \frac{\frac{0}{k^{10}}}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0 + h, 0) - v(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^4}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0, 0 + k) - v(0, 0)}{k} = \lim_{k \to 0} \frac{\frac{0}{k^{10}}}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

From the above results, it is clear that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C-R equations are satisfied at the origin.

But $f'(0) = \lim_{z \to 0} \frac{f(0+z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{x^2 y^5 (x+iy)}{x^4 + y^{10}} - 0 \right] \cdot \frac{1}{x+iy}$ (Increment = z)

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y^5}{x^4 + y^{10}}$$

Let $z \to 0$ along the radius vector $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{m^5 x^7}{x^4 + m^{10} x^{10}} = \lim_{x \to 0} \frac{m^5 x^3}{1 + m^{10} x^6} = \frac{0}{1} = 0 \quad \text{... (1)}$$

Again let $z \to 0$ along the curve $y^5 = x^2$

$$f'(0) = \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2} \quad \text{... (2)}$$

(1) and (2) shows that $f'(0)$ does not exist. Hence, $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there. **Ans.**

10 C–R EQUATIONS IN POLAR FORM

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

Proof. We know $x = r \cos \theta$, and u is a function of x and y.

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$u + iv = f(z) = f(re^{i\theta}) \quad \text{... (1)}$$

Differentiating (1) partially w.r.t., "r", we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}).e^{i\theta} \quad \text{... (2)}$$

Differentiating (1) w.r.t. "θ", we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})re^{i\theta}i \quad \text{... (3)}$$

Substituting the value of $f'(re^{i\theta})e^{i\theta}$ from (2) in (3), we obtain

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)i \quad \text{or} \quad \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Equating real and imaginary parts, we get

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}} \Rightarrow \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

And $$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$ **Proved.**