Theory of Scattering of Electromagnetic Waves

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1 Scattering and Scattering Parameters

Whenever an electromagnetic wave is directed at matter, the electrons of the matter are set into periodic motion due to periodically vibrating electric and magnetic fields of the incident electromagnetic wave. Since a periodically vibrating electron is equivalent to an accelerated electron and an accelerated charge always radiates out energy, therefore the electron in turn emits electromagnetic wave into space. This phenomenon is known as *scattering* and re-emitted electromagnetic wave is known as scattered radiation.

1.1 Elastic Scattering

If the energies of the incident and scattered radiation are equal, the phenomenon is known as *elastic scattering*.

1.2 Inelastic Scattering

If the energies of the incident and scattered radiation are different, the phenomenon is known as *inelastic scattering*.

The scattering process is described in terms of a quantity known as *scattering cross-section*. To do this let us note that each molecule of the matter presents to the incident radiation a target area of πd^2 where d is the diameter of the molecule. This target area is just a cross-section of the region within which an interaction of electromagnetic wave and molecule of the target can take place as viewed along the direction of incident beam. This is where the name scattering cross-section comes from. Now we shall define differential and total scattering cross-section.

1.3 Differential Scattering Cross-Section

The differential scattering cross-section in any direction is defined as the ratio of the amount of energy scattered by target in this direction per unit solid angle per second to the incident energy flux density (i.e. incident energy per unit area normal to the direction of incident radiation) per second i.e.

$$\sigma(\Omega) = \frac{\text{amount of energy scattered by target in this direction per unit solid angle per second}}{\text{incident energy flux density per second}} \tag{1}$$

Let $d\Omega$ be the solid angle subtended by an element of area da at the scatterer

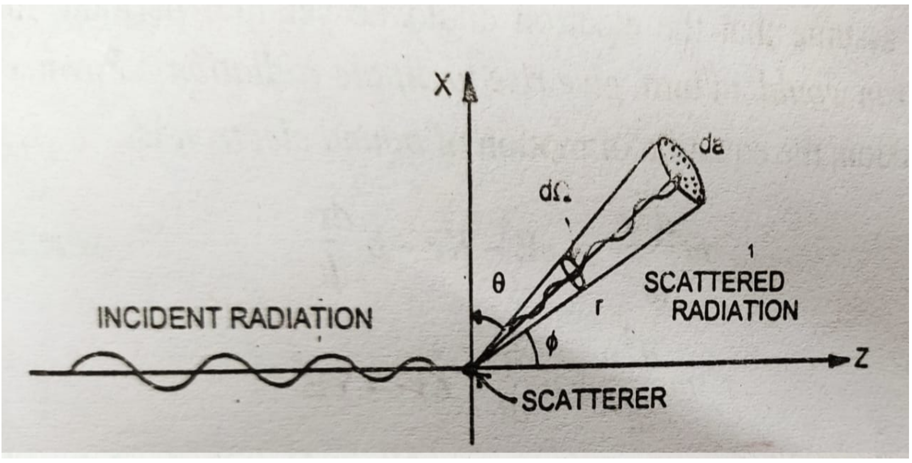

Figure 1:

as shown in Fig 1. If S_S is the intensity of the scattered radiation, then amount of energy scattered per second in the direction $d\Omega$ is given by

$$dP_S = S_S da \tag{2}$$

$\therefore$ then amount of energy scattered per second per unit solid angle is given by

$$\frac{dP_S}{d\Omega} = \frac{S_S da}{d\Omega} = S_S r^2 \quad \left(\because d\Omega = \frac{da}{r^2}\right) \tag{3}$$

Hence the differential scattering cross-section $\sigma(\Omega)$ is given by

$$\begin{aligned} \sigma(\Omega) &= \frac{\text{energy scattered per second per unit solid angle } (dP_S/d\Omega)}{\text{incident energy flux density per second } S_i} \\ &= \frac{S_S r^2}{S_i} \end{aligned} \tag{4}$$

From equation (4) it is clear that the differential scattering cross-section has the dimensions of cross-section (area).

1.4 Total scattering cross-section

The total scattering cross-section (σ_T) is obtained by integrating differential cross-section over all angles i.e.

$$\begin{aligned} \sigma_T &= \int \sigma(\Omega) d\Omega = \int \frac{S_S}{S_i} r^2 d\Omega \quad \text{(using equation 4)} \\ &= \int \frac{S_S}{S_i} r^2 . \frac{da}{r^2} \\ &= \int \frac{S_S}{S_i} da = \frac{P_S}{S_i} \end{aligned} \tag{5}$$

where $P_S = \int S_S da =$ total energy scattered per second (or total power scattered).

Thus *the total scattering cross-section σ_T is defined as the ratio of total power (i.e. energy per second) scattering in all directions to the intensity (i.e. energy per unit area per second) of the incident radiations.*

2 Theory of scattering

When an Electromagnetic wave passes through a medium, the atomic (or molecular) dipoles are created in the medium due to relative displacement of electrons and nuclei of neutral atoms (or molecules) due to the electric field of the wave.
The net force of charge q is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{6}$$

In a plane Electromagnetic wave

$$\mathbf{B} = \frac{\mathbf{n} \times \mathbf{E}}{c} \tag{7}$$

Hence, equation (1) becomes

$$\mathbf{F} = q\left[\mathbf{E} + \frac{\mathbf{v}}{\mathbf{c}} \times (\mathbf{n} \times \mathbf{E})\right] = q\mathbf{E} \quad (\text{as} \quad \mathbf{v}/\text{c} \ll 1) \tag{8}$$

Thus a dipole moment is induced in the atom. These atomic dipoles execute forced vibrations under the action of the oscillatory electric field of the incident wave. Since nucleus is heavy, we may assume that the electron could be set into periodic motion by the field. The periodic motion of the electron would, in turn, give rise to dipole radiation; known as scattered radiation.

As in the case of dispersion, the equation of motion of bound electron is

$$m\frac{d^2\mathbf{r}}{dt^2} = e\mathbf{E} - K\mathbf{r} - b\frac{d\mathbf{r}}{dt} \tag{9}$$

where $\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}.\mathbf{r}-i\omega t}$.
Or,

$$m\frac{d^2\mathbf{r}}{dt^2} + b\frac{d\mathbf{r}}{dt} + K\mathbf{r} = e\mathbf{E} \tag{10}$$

or

$$\frac{d^2\mathbf{r}}{dt^2} + \frac{b}{m}\frac{d\mathbf{r}}{dt} + \frac{K}{m}\mathbf{r} = \frac{e}{m}\mathbf{E} \tag{11}$$

Substituting $\frac{b}{m} = g$ and $\frac{K}{m} = \omega_0^2$, we get

$$\frac{d^2\mathbf{r}}{dt^2} + g\frac{d\mathbf{r}}{dt} + \omega_0^2\mathbf{r} = \frac{e}{m}\mathbf{E} \tag{12}$$

This equation represents the equation of damped harmonic oscillator in forced oscillation with ω_0 as natural frequency of oscillator.

From equation (12) it is clear that **E** and **r** are along the same direction; hence we may use magnitudes of vectors and write equation (12) as

$$\frac{d^2r}{dt^2} + g\frac{dr}{dt} + \omega_0^2 r - \frac{e}{m}E = 0 \tag{13}$$

Now let us consider that the incident electromagnetic wave is a plane polarised having its electric vector along X-axis and is propagating along Z-direction; then acceleration of the charge (e) will be along X-axis (i.e. along **E**) and

$$\because \mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}.\mathbf{r}-i\omega t} \tag{14}$$

Or

$$E = E_0 e^{-i(\omega t - \mathbf{k}.\mathbf{r})} = E_0 e^{-i(\omega t - kz)} \tag{15}$$

so that the equation of motion of electron (equation (13)) may be written as

$$\frac{d^2x}{dt^2} + g\frac{dx}{dt} + \omega_0^2 x = \frac{e}{m}E_0 e^{-i(\omega t - kz)} \tag{16}$$

The solution of the above equation is given by

$$x = \frac{[e/m]E_0 e^{-i(\omega t - kz)}}{\omega_0^2 - \omega^2 - ig\omega} \tag{17}$$

Multiplying and dividing by $(\omega_0^2 - \omega^2 + ig\omega)$, we get

$$\begin{aligned} x &= \frac{[e/m](\omega_0^2 - \omega^2 + ig\omega)E_0 e^{-i(\omega t - kz)}}{(\omega_0^2 - \omega^2)^2 + g^2\omega^2} \\ &= \frac{[eE_0/m]e^{-i(\omega t - kz)}}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]^{1/2}}\left[\frac{\omega_0^2 - \omega^2}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]^{1/2}} + i\frac{g\omega}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]^{1/2}}\right] \\ &= \frac{[eE_0/m]e^{-i(\omega t - kz)}}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]^{1/2}}(\cos\delta + i\sin\delta) \end{aligned} \tag{18}$$

where

$$\cos\delta = \frac{\omega_0^2 - \omega^2}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]^{1/2}} \tag{19}$$

and

$$\sin\delta = \frac{g\omega}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]^{1/2}} \tag{20}$$

So that

$$\begin{aligned} \tan\delta &= \frac{\sin\delta}{\cos\delta} \\ &= \frac{g\omega}{\omega_0^2 - \omega^2} \end{aligned} \tag{21}$$

So equation (18) becomes

$$\begin{aligned} x &= \frac{\left[\frac{eE_0}{m}\right] e^{-i(\omega t - kz)}}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]^{1/2}} e^{i\delta} \\ &= \frac{\left[\frac{eE_0}{m}\right] e^{-i(\omega t - kz - \delta)}}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]^{1/2}} \end{aligned} \tag{22}$$

Now as oscillatory electron is equivalent to an induced oscillating electric dipole of moment

$$p = ex \tag{23}$$

Substituting value of x from equation (22), we get

$$\begin{aligned} p &= \frac{\left[\frac{e^2E_0}{m}\right] e^{-i(\omega t - kz - \delta)}}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]^{1/2}} \\ &= |p_0| e^{-i(\omega t - kz - \delta)} \end{aligned} \tag{24}$$

where

$$|p_0| = \frac{\left[\frac{e^2E_0}{m}\right]}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]^{1/2}} \tag{25}$$

But the *average energy radiated per second per unit area in a normal direction by an oscillating dipole* is given by

$$\begin{aligned} S_S &= \frac{1}{4\pi\epsilon_0} \frac{\omega^4 |p_0|^2}{8\pi c^3 r^2} \sin^2\theta \\ &= \frac{1}{4\pi\epsilon_0} \frac{\omega^4}{8\pi c^3 r^2} \frac{\left[\frac{e^2E_0}{m}\right]^2 \sin^2\theta}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]} \quad \text{(using eqn(25))} \\ &= \frac{1}{4\pi\epsilon_0} \frac{\omega^4}{8\pi c^3 r^2} \frac{\left[e^4 E_0^2/m^2\right] \sin^2\theta}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2\right]} \end{aligned} \tag{26}$$

Further for a plane wave the incident intensity is given by Poynting vector

$$\begin{aligned} \mathbf{S_i} &= \mathbf{E} \times \mathbf{H} = \frac{1}{2} E_0 H_0 \\ &= \frac{1}{2} \epsilon_0 c E_0^2 \end{aligned} \quad (27)$$

$$\left[\text{as} \quad \text{H}_0 \sqrt{\frac{\epsilon_0}{\mu_0}} \text{E}_0 = (\text{c}\epsilon_0 \text{E}_0) \right] \quad (28)$$

So the *differential scattering cross-section*

$$\begin{aligned} \sigma(\Omega) &= \frac{S_S}{S_i} r^2 = \frac{\frac{1}{4\pi\epsilon_0} \frac{\omega^4}{8\pi c^3 r^2} \left[(e^4 E_0^2/m^2) \sin^2\theta \right] / \left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2 \right]}{\frac{1}{2}\epsilon_0 c E_0^2} \\ &= \left[\frac{e^2}{4\pi\epsilon_0 m c^2} \right]^2 \frac{\omega^4 \sin^2\theta}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2 \right]} \\ &= r_0^2 \frac{\omega^4 \sin^2\theta}{\left[(\omega_0^2 - \omega^2)^2 + g^2\omega^2 \right]} \end{aligned} \quad (29)$$

where $r_0 = \frac{e^2}{4\pi\epsilon_0 m c^2}$ is known as the *classical radius of the electron.*

Sometimes it is desirable to express the differential scattering cross-

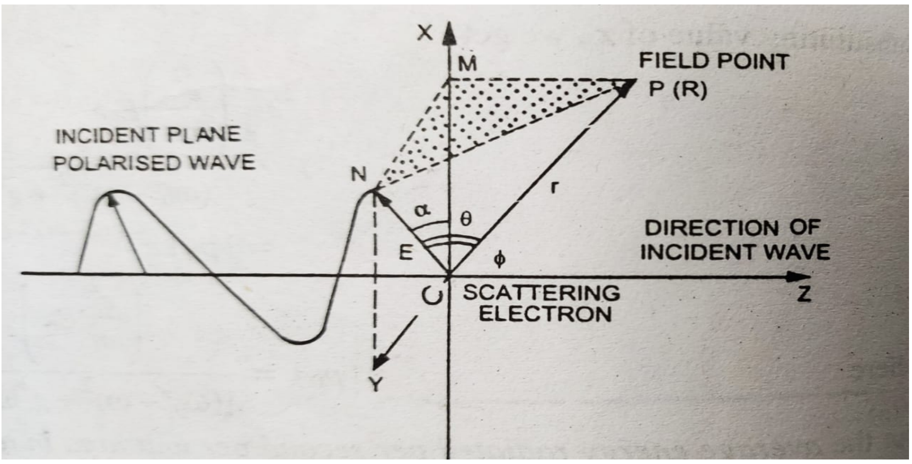

Figure 2:

section in terms of angle of scattering ϕ and polarisation angle α shown in the Fig.2 instead of angle θ which is the angle between $\mathbf{E}$ and $\mathbf{r}$. To find a relation between α, ϕ and θ, we draw a plane PMN perpendicular to plane containing $\mathbf{E}$, then ON=$r\cos\theta$.

OM=$r\sin\phi$ and again ON=OM$\cos\alpha = r\sin\phi\cos\alpha$; thereby giving

$$\cos\theta = \sin\phi\cos\alpha \quad (30)$$

or

$$\cos^2\theta = \sin^2\phi\cos^2\alpha \tag{31}$$

$$\begin{aligned} \because \sin^2\theta &= 1-\cos^2\theta \\ &= 1-\sin^2\phi\cos^2\alpha \\ &= 1-\cos^2\alpha(1-\cos^2\phi). \end{aligned} \tag{32}$$

For *plane polarised wave* $\alpha = 0$;

$$\therefore \sin^2\theta = \cos^2\phi \tag{33}$$

If the *primary wave is unpolarised (i.e. unpolarised)*, we must average over θ, thus we obtain

$$\begin{aligned} \bar{\sin^2}\theta &= 1-\bar{\cos^2}\alpha(1-\cos^2\phi) \\ &= 1-\frac{1}{2}(1-\cos^2\phi) \qquad (\because \bar{\cos^2}\alpha = \frac{1}{2}) \\ &= \frac{1}{2}(1+\cos^2\phi) \end{aligned} \tag{34}$$

Making this substitution, equation (29) gives

$$\sigma(\Omega) = \frac{r_0^2\omega^4}{\left[(\omega_0^2-\omega^2)^2+g^2\omega^2\right]}\left\{\frac{1}{2}(1+\cos^2\phi)\right\} \tag{35}$$

The factor $\frac{1}{2}(1+\cos^2\phi)$ is the *polarisation factor known as* degree of depolarisation. From equation (35) it is clear that

1. Scattering depends on the frequency ω of the incident radiation.
2. Scattering depends on the angle of scattering ϕ
3. Scattering depends on the nature of scatterer (due to factors ω_0 and g).

The total *scattering cross-section* is obtained by integrating differential scattering cross-section over all angle i.e.

$$\begin{aligned} \sigma_T &= \int \sigma(\Omega)d\Omega \\ &= \frac{r_0^2\omega^4}{\left[(\omega_0^2-\omega^2)^2+g^2\omega^2\right]}\int_0^\pi frac12(1+\cos^2\phi)2\pi\sin\phi d\phi \\ &= \frac{r_0^2\omega^4}{\left[(\omega_0^2-\omega^2)^2+g^2\omega^2\right]}\cdot\frac{8}{3}\pi \\ &= \frac{8}{3}\pi\frac{r_0^2\omega^4}{\left[(\omega_0^2-\omega^2)^2+g^2\omega^2\right]} \end{aligned} \tag{36}$$

This equation shows that total scattering cross-section, in general, depends upon the frequency ω of incident radiation. Fig. 3 represents a plot of σ_T against ω.

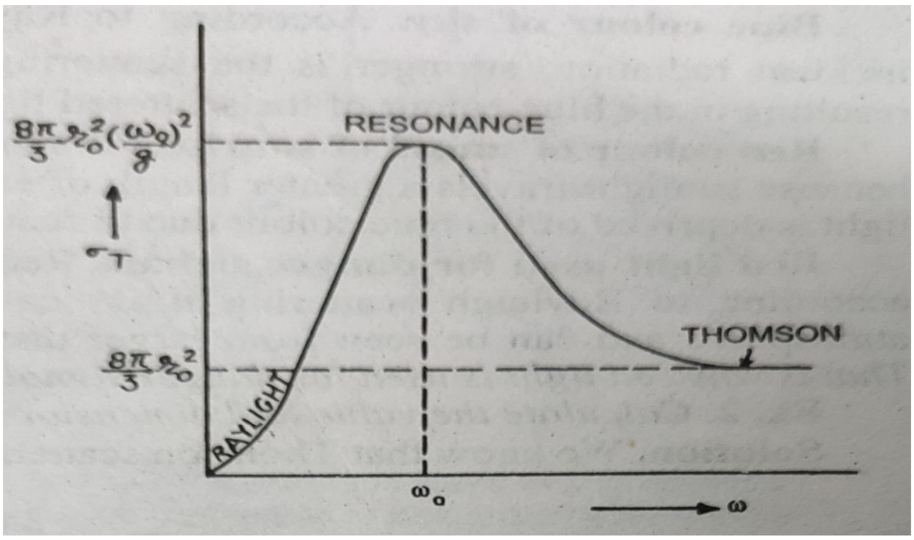

Figure 3: Total scattering cross-section σ_T against the frequency ω of incident radiation