Dr. ASHOK KUMAR

Guest faulty Department of Physics

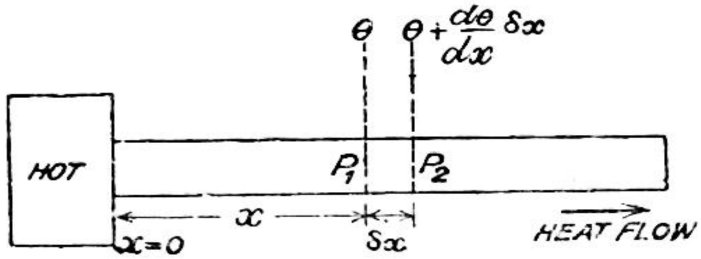
Magadh Mahila Collage, P.U.

Name of Program –Physics (Hons), Part I

Paper –II , Group-A

Rectilliner Flow of Heat in a Metal Rod

Consider a long metal bar of uniform area of cross-section placed parallel to x-axis and heated at one end.



Consider two parallel planes P_1 and P_2 perpendicular to the length of the bar at distance x and $x+\delta x$ from the hot end. Let θ be the excess of temperature above surroundings at the plane P_1 and $\frac{d\theta}{dx}$ the temperature gradient. Then

Excess of temperature at $P_2 = \theta + \frac{d\theta}{dx} . \delta x$

and temperature gradient at $P_2 = \frac{d}{dx}\left(\theta + \frac{d\theta}{dx}\delta x\right)$.

If Q_1 and Q_2 are respectively the quantities of heat entering the plane at P_1 and leaving the plane at P_2 per second, then

$$Q_1 = -KA\frac{d\theta}{dx}$$

where A is the cross-sectional area of the bar and K the thermal conductivity of its material,

and $$Q_2 = -KA\frac{d}{dx}\left(\theta + \frac{d\theta}{dx}\delta x\right).$$

Therefore net gain of heat per second by the section δx between P_1 and P_2 of the rod

$$Q = Q_1 - Q_2 = -KA \frac{d\theta}{dx} - \left\{ -KA \frac{d}{dx}\left(\theta + \frac{d\theta}{dx} . \delta x \right)\right\}$$

$$Q = KA \frac{d^2\theta}{dx^2} \delta x. \quad \text{...(i)}$$

CASE I.

Before the steady state is reached. The amount of heat Q is partly used up in raising the temperature of the section and the rest is lost to the surroundings due to radiations. Let ρ be the density and S the specific heat of the material of bar, and $d\theta/dt$ be the rate of rise of temperature.

=mass of the section × specific heat × rise in temp. per sec.

$$=(A\ \delta x)\ \rho \times S \times \frac{d\theta}{dt} \cdot \quad \text{...(ii)}$$

If E is the **emissivity** of the surface, *i.e.*, the heat lost per second per unit area of the surface per unit temperature excess over the surroundings, and p the perimeter of the cross-section of bar, Heat lost per second due to radiations.

=emissivity × surface area of the section × temp. excess

$$= E \times p . \delta x = \theta. \quad \text{..(iii)}$$

Hence from (i), (ii) and (iii)

$$Q = KA \frac{d^2\theta}{dx^2}\ \delta x = A\ \delta x\ \rho \times S \times \frac{d\theta}{dt} + E \times p \delta x \times \theta .$$

Dividing throughout by $KA\ \delta x$, we get

$$\frac{d^2\theta}{dx^2} = \frac{\rho S}{K} \cdot \frac{d\theta}{dt} + \frac{Ep}{KA} \theta. \quad \text{...(iv)}$$

This is the general equation representing the unidirectional rectilinear flow of heat and is known as Fourier's differential equation.

If the heat lost by radiations is negligible, *i.e.*, the bar is perfectly, insulated from surroundings, then its emissivity $E=0$ and eqn. (iv) reduces to

$$\frac{d^2\theta}{dx^2} = \frac{\rho S}{K} \cdot \frac{d\theta}{dt}$$

or

$$\frac{d\theta}{dt} = \frac{K}{\rho S} \cdot \frac{d^2\theta}{dx^2} = h \frac{d^2\theta}{dx^2} \quad \text{.. (v)}$$

where $K/\rho S = h$ is the **thermal diffusivity,** which determines the rate at which temperature changes take place in the bar.

CASE II.

After the steady state is reached. The temperature at every point of the bar becomes stationary, *i.e.*, $d\theta/dt = 0$.

then equation (iv) reduces to

$$\frac{d^2\theta}{dx^2} = \frac{Ep}{KA} \cdot \theta = \mu^2\theta \quad \text{...(vi)}$$

where
$$\frac{EP}{KA} = \mu^2.$$

In order to solve this differential equation, let us put

$$\theta = ae^{\alpha x} \quad \text{...(vii)}$$

where a and α are constants. Differentiating it twice, we have

$$\frac{d\theta}{dx} = a\alpha.e^{\alpha x} \text{ and } \frac{d^2\theta}{dx^2} = \alpha^2.ae^{\alpha x}$$

$$\therefore \quad \alpha^2.ae^{\alpha x} = \mu^2.ae^{\alpha x}$$

or
$$\alpha^2 = \mu^2$$

or
$$\alpha = \pm\mu.$$

Substituting in eqn. (vii), the value of α, we get the solution of eqn (vi) to be

$$\theta = ae^{\pm\mu x}.$$

Hence the most general solution can be written as

$$\theta = a_1e^{+\mu x} + a_2e^{-\mu x} \quad \text{...(viii)}$$

If the bar is supposed to be infinitely long, its cold end will be nearly at the temperature of the surroundings, *i.e.* at $x = \infty$, $\theta = 0$. Also if θ_0 is the excess temperature of the hot end above surroundings, then at $x = 0$, $\theta = 0_0$. Applying the first condition, we have from equation (viii)

$$0 = a_1e^{+\mu\infty} + a_2e^{-\mu\infty}$$

or
$$0 = a_1e^{+\mu\infty} \quad (\because \quad e^{-\mu\infty} = 0).$$

$$\therefore \quad a_1 = 0 \qquad (\because \quad e^{-\mu} = 1).$$

Applying the second condition (at $x = 0$, $\theta = \theta_0$) and putting $a_1 = 0$ in eqn. (viii), we get

$$\theta_0 = 0 + a_2e^{-\mu_0}$$

or
$$a_2 = \theta_0 \qquad (\because \quad e^{-\mu_0} = 1)$$

Substituting the values of a_1 and a_2 in eqn. (viii), it becomes

$$\theta = \theta_0 \; e^{-\mu x}. \quad \text{...(ix)}$$

This equation gives the excess of temperature of a point at a distance x from the hot end along the bar.