GAUSS DIVERGENCE THEOREM, STOKES' THEOREM, and GREEN'S THEOREM

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1 THE DIVERGENCE THEOREM OF GAUSS

The divergence theorem of Gauss states that if V is the volume bounded by a closed surface S and **A** is a vector function of position with continuous derivatives, then

$$\iiint_V \nabla.\mathbf{A} dV = \iint_S \mathbf{A}.\hat{\mathbf{n}} dS = \oiint_S \mathbf{A}.d\mathbf{S} \qquad (1)$$

where $\hat{\mathbf{n}}$ is the positive (outward drawn) normal to S.

Gauss diveregnce theorem can also be stated as following:

The surface integral of the normal component of a vector **A** taken over a closed surface is equal to the integral of the divergence of **A** taken over the volume enclosed by the surface.

Proof: Let S be a closed surface which is such that any line parallel to

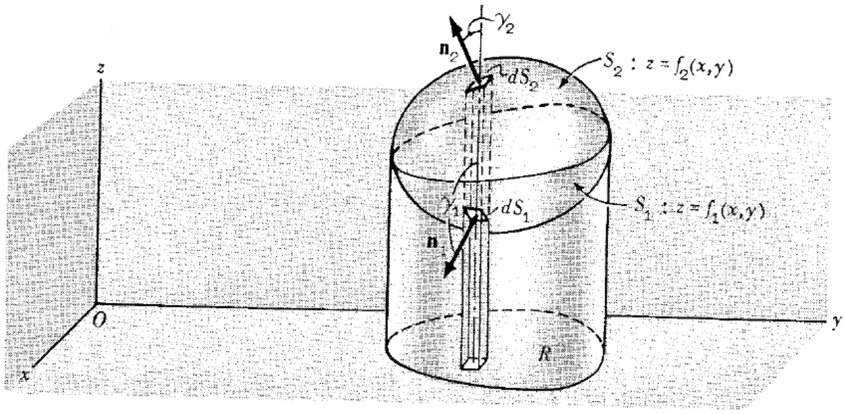

Figure 1:

the coordinate axes cuts S in at most two points. Assume the equations of the lower and upper portions, S_1 and S_2, to be $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. Denote the projection of the surface on the xy plane by R. Consider

$$\begin{aligned}
\iiint_V \frac{\partial A_3}{\partial z} dV &= \iiint_V \frac{\partial A_3}{\partial z} dz dy dx \\
&= \iint_R \left[\int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial A_3}{\partial z} dz \right] dy dx \\
&= \iint_R A_3(x, y, z)|_{z=f_1}^{f_2} dy dx \\
&= \iint_R \left[A_3(x, y, f_2) - A_3(x, y, f_1) \right] dy dx \qquad (2)
\end{aligned}$$

For the upper portion S_2, $dydx = \cos\gamma_2 dS_2 = \mathbf{k}.\mathbf{n_2} dS_2$ since the normal $\mathbf{n_2}$ to S_2 makes an acute angle γ_2 with $\mathbf{k}$.

For the lower portion S_1, $dydx = -\cos\gamma_1 dS_2 = \mathbf{k}.\mathbf{n_1} dS_1$ since the normal $\mathbf{n_1}$ to S_1 makes an obtuse angle γ_1 with $\mathbf{k}$.

Then

$$\begin{aligned}
\iint_R A_3(x, y, f_2) dy dx &= \iint_{S_2} A_3 \mathbf{k}.\mathbf{n_2} dS_2 \\
\iint_R A_3(x, y, f_1) dy dx &= -\iint_{S_1} A_3 \mathbf{k}.\mathbf{n_1} dS_1
\end{aligned}$$

and

$$\begin{aligned}
\iint_R A_3(x, y, f_2) dy dx - \iint_R A_3(x, y, f_1) dy dx &= \iint_{S_2} A_3 \mathbf{k}.\mathbf{n_2} dS_2 + \iint_{S_1} A_3 \mathbf{k}.\mathbf{n_1} dS_1 \\
&= \iint_S A_3 \mathbf{k}.\mathbf{n} dS \qquad (3)
\end{aligned}$$

so from equations (2) and (3)

$$\iiint_V \frac{\partial A_3}{\partial z} dV = \iint_S A_3 \mathbf{k}.\mathbf{n} dS \qquad (4)$$

Similarly, by projecting S on the other coordinate planes,

$$\begin{aligned}
\iiint_V \frac{\partial A_1}{\partial x} dV &= \iint_S A_1 \mathbf{i}.\mathbf{n} dS \qquad (5) \\
\iiint_V \frac{\partial A_2}{\partial y} dV &= \iint_S A_2 \mathbf{j}.\mathbf{n} dS \qquad (6)
\end{aligned}$$

Adding equations (4), (5) and (6),

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV = \iint_S \left(A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k} \right).\mathbf{n} dS \qquad (7)$$

Or

$$\iiint_V \nabla.\mathbf{A}dV = \iint_S \mathbf{A}.\hat{\mathbf{n}}dS = \oiint_S \mathbf{A}.d\mathbf{S} \qquad (8)$$

Proved.

1.1 Physical demonstration of the divergence theorem

Let $\mathbf{A}$ = velocity $\mathbf{v}$ at any point of a moving fluid. From Figure (2 a) below:

$$\begin{aligned} \text{Volume of fluid} \quad & \text{crossing dS in } \Delta t \text{seconds} \\ = \; & \text{volume contained in cylinder of base dS and slant height} \mathbf{v}\Delta t \\ = \; & (\mathbf{v}\Delta t).\mathbf{n}dS = \mathbf{v}.\mathbf{n}dS\Delta t \end{aligned}$$

Then, volume per second of fluid crossing dS = $\mathbf{v}.dS$ Prom Figure (2 b)

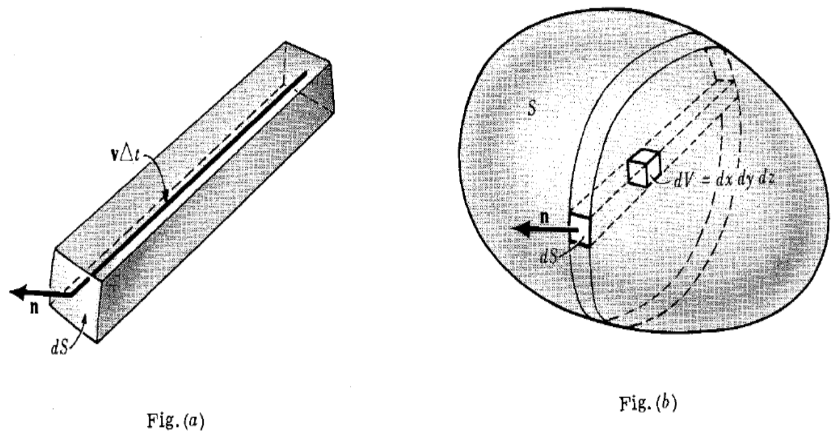

Figure 2:

above:

Total volume per second of fluid emerging from closed surface S

$$= \iint_S \mathbf{v}.\mathbf{n}dS$$

And we know that $\nabla.\mathbf{v}dV$ is the volume per second of fluid emerging from a volume element dV. Then

Total volume per second of fluid emerging from all volume elements in S

$$= \iiint_V \nabla.\mathbf{v}dV$$

Thus

$$\iint_S \mathbf{v}.\mathbf{n}dS = \iiint_V \nabla.\mathbf{v}dV \tag{9}$$

2 STOKES' THEOREM

Stokes' theorem states that if S is an open, two-sided surface bounded by a closed, nonintersecting curve C (simple closed curve) then if **A** has continuous derivatives

$$\oint_C \mathbf{A}.d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) .\hat{\mathbf{n}}dS = \iint_S (\nabla \times \mathbf{A}) .d\mathbf{S} \tag{10}$$

where C is traversed in the positive direction. The direction of C is called positive if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of the positive normal to S, has the surface on his left.

In other words Stokes' theorem may be stated as following:
The line integral of the tangential component of a vector **A** taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of **A** taken over any surface S having C as its boundary.

Proof: Let S be a surface which is such that its projections on the xy,

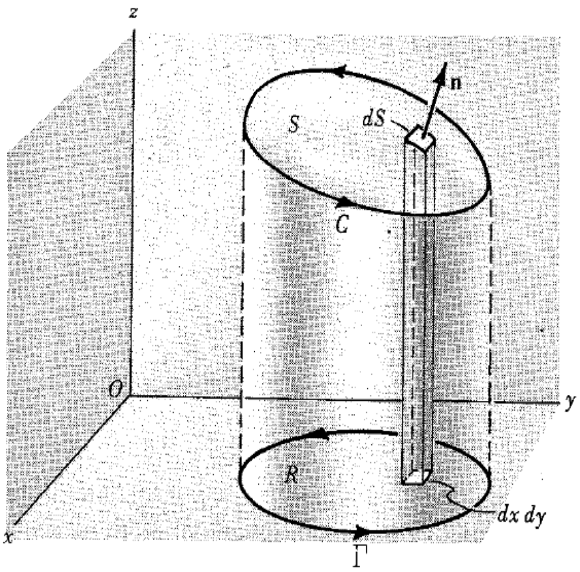

Figure 3:

yz and xz planes are regions bounded by simple closed curves, as indicated in the adjoining figure. Assume S to have representation $z = f(x, y)$ or $x = g(y, z)$ or $y = h(x, z)$, where f, g, h are single-valued, continuous and differentiable functions. We must show that

$$\begin{aligned}\iint_S (\nabla \times \mathbf{A}) . d\mathbf{S} &= \iint_S [\nabla \times (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})] . \mathbf{n} dS \\ &= \oint_C \mathbf{A} . d\mathbf{r} \qquad (11)\end{aligned}$$

where Cis the boundary of S.

Consider first $\iint_S [\nabla \times A_1\mathbf{i}] . \mathbf{n} dS$.

Since

$$\nabla \times (A_1\mathbf{i}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix} = \frac{\partial A_1}{\partial z}\mathbf{j} - \frac{\partial A_1}{\partial y}\mathbf{k},$$

$$[\nabla \times A_1\mathbf{i}] . \mathbf{n} dS = \left(\frac{\partial A_1}{\partial z}\mathbf{n}.\mathbf{j} - \frac{\partial A_1}{\partial y}\mathbf{n}.\mathbf{k} \right) dS \qquad (12)$$

If $z = f(x, y)$ is taken as the equation of S, then the position vector to any point of S is $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$ so that $\frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$. But $\frac{\partial r}{\partial y}$ is a vector tangent to S and thus perpendicular to $\mathbf{n}$, so that

$$\mathbf{n}.\frac{\partial \mathbf{r}}{\partial y} = \mathbf{n}.\mathbf{j} + \frac{\partial z}{\partial y}\mathbf{n}.\mathbf{k} = 0 \qquad (13)$$

Or

$$\mathbf{n}.\mathbf{j} = -\frac{\partial z}{\partial y}\mathbf{n}.\mathbf{k} \qquad (14)$$

Substituting equation (14) into equation (12), we obtain

$$\left(\frac{\partial A_1}{\partial z}\mathbf{n}.\mathbf{j} - \frac{\partial A_1}{\partial y}\mathbf{n}.\mathbf{k} \right) dS = \left(-\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\mathbf{n}.\mathbf{k} - \frac{\partial A_1}{\partial y}\mathbf{n}.\mathbf{k} \right) dS \qquad (15)$$

Or

$$[\nabla \times A_1\mathbf{i}] . \mathbf{n} dS = -\left(\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial A_1}{\partial y} \right) \mathbf{n}.\mathbf{k} dS \qquad (16)$$

Now on S,$_1(x, y, z) = A_1(x, y, f(x, y)) = F(x, y)$; hence $\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial A_1}{\partial y} = \frac{\partial F}{\partial y}$ and equation (16) becomes

$$[\nabla \times A_1\mathbf{i}] . \mathbf{n} dS = -\frac{\partial F}{\partial y}\mathbf{n}.\mathbf{k} dS = -\frac{\partial F}{\partial y} dx dy \qquad (17)$$

Then

$$\iint_S [\nabla \times A_1\mathbf{i}] . \mathbf{n} dS = -\iint_R \frac{\partial F}{\partial y} dx dy \qquad (18)$$

where R is the projection of S on the xy plane. By Green's theorem for the plane the last integral equals $\oint_\Gamma F dx$ where Γ is the boundary of R. Since at each point (x, y) of Γ the value of F is the same as the value of A_1 at each point (x, y, z) of C, and since dx is the same for both curves, we must have

$$\oint_\Gamma F dx = \oint_C A_1 dx \tag{19}$$

or

$$\iint_S [\nabla \times A_1 \mathbf{i}] \,.\mathbf{n} dS = \oint_C A_1 dx \tag{20}$$

Similarly, by projections on the other coordinate planes,

$$\iint_S [\nabla \times A_1 \mathbf{i}] \,.\mathbf{n} dS = \oint_C A_1 dx \tag{21}$$

$$\iint_S [\nabla \times A_2 \mathbf{j}] \,.\mathbf{n} dS = \oint_C A_2 dy \tag{22}$$

$$\iint_S [\nabla \times A_3 \mathbf{k}] \,.\mathbf{n} dS = \oint_C A_3 dz \tag{23}$$

Thus by addition,

$$\iint_S [\nabla \times \mathbf{A}] \,.\mathbf{n} dS = \oint_C \mathbf{A}.\mathbf{dr} \tag{24}$$

Proved.

The theorem is also valid for surfaces S which may not satisfy the restrictions imposed above. For assume that S can be subdivided into surfaces $S_1, S_2, ...S_k$ with boundaries $C_1, C_2, ...C_k$ which do satisfy the restrictions. Then Stokes' theorem holds for each such surface. Adding these surface integrals, the total surface integral over S is obtained. Adding the corresponding line integrals over $C_1, C_2, ...C_k$, the line integral over is obtained.

3 GREEN'S THEOREM IN THE PLANE

If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \tag{25}$$

where C is traversed in the positive (counterclockwise) direction. Unless otherwise stated we shall always assume $\oint$ to mean that the integral is described in the positive sense.

Green's theorem in the plane is a special case of Stokes' theorem. Also, it is of interest to notice that Gauss' divergence theorem is a generalization of Green's theorem in the plane where the (plane) region R and its closed boundary (curve) C are replaced by a (space) region V and its closed boundary (surface) S. For this reason the divergence theorem is often called *Green's theorem in space.*

Green's theorem in the plane also holds for regions bounded by a finite number of simple closed curves which do not intersect.

Proof: Let the equations of the curves AEB and AFB (see Fig. 4) be

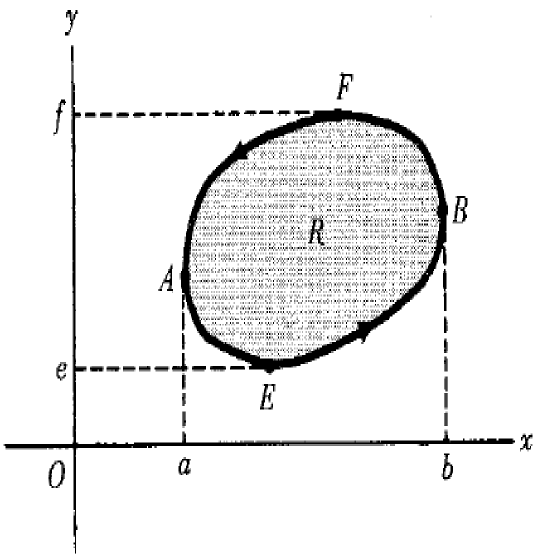

Figure 4:

$y = Y_1(x)$ and $y = Y_2(x)$ respectively. If R is the region bounded by C, we have

$$\begin{aligned}
\iint_R \frac{\partial M}{\partial y} dxdy &= \int_{x=a}^{x=b} \left[\int_{y=Y_1(x)}^{y=Y_2(x)} \frac{\partial M}{\partial y} dy \right] dx \\
&= \int_{x=a}^{x=b} M(x,y)|_{y=Y_1(x)}^{y=Y_2(x)} dx \\
&= \int_a^b \left[M(x,Y_2) - M(x,Y_1) \right] dx \\
&= -\int_a^b M(x,Y_1)dx - \int_b^a M(x,Y_2)dx = -\oint_C Mdx \quad (26)
\end{aligned}$$

Then

$$\oint_C Mdx = -\iint_R \frac{\partial M}{\partial y} dxdy \quad (27)$$

Similarly let the equations of curves EAF and EBF be $x = X_1(y)$ and $x = X_2(y)$ respectively. Then

$$\begin{aligned}
\iint_R \frac{\partial N}{\partial x} dxdy &= \int_{y=e}^{x=f} \left[\int_{x=X_1(y)}^{x=X_2(y)} \frac{\partial N}{\partial x} dx \right] dy \\
&= \int_{y=e}^{y=f} N(x,y)|_{x=X_1(y)}^{x=X_2(y)} dy \\
&= \int_e^f \left[N(X_2,y) - N(X_1,y) \right] dy \\
&= \int_f^e N(X_1,y)dy + \int_e^f N(X_2,y)dy = \oint_C Ndy \quad (28)
\end{aligned}$$

Then

$$\oint_C Ndy = \iint_R \frac{\partial N}{\partial x} dxdy \quad (29)$$

Adding equations (27) and (29)

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy. \quad (30)$$

Proved.