The Continuity Equation and Poynting's Theorem

Dr. Priti Mishra

1 The Continuity Equation

In this chapter we study conservation of energy in electrodynamics. But I want to begin by reviewing the conservation of charge, because it is the paradigm for all conservation laws. What precisely does conservation of charge tell us? That the total charge in the universe is constant? Well, sure-that's global conservation of charge; but local conservation of charge is a much stronger statement: If the total charge in some volume changes, then exactly that amount of charge must have passed in or out through the surface.

Formally, the charge in a volume V is

$$Q(t) = \int_V \rho_(\mathbf{r}, t) d\tau, \tag{1}$$

and the current flowing out through the boundary S is $\int_S \mathbf{J}.\mathbf{da}$ so local conservation of charge says

$$\frac{dQ}{dt} = -\int_S \mathbf{J}.\mathbf{da} \tag{2}$$

Using Equation (1) to rewrite the left side, and invoking the divergence theorem on the right, we have

$$\int_V \frac{\partial \rho(\mathbf{r}, t)}{\partial t} d\tau = -\int_V \nabla.\mathbf{J} d\tau \tag{3}$$

or

$$\int_V \left(\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla.\mathbf{J} \right) d\tau = 0 \tag{4}$$

and since this is true for any volume, it follows that

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla.\mathbf{J} = 0 \tag{5}$$

or

$$\boxed{\frac{\partial \rho(\mathbf{r},t)}{\partial t} = -\nabla.\mathbf{J} \qquad (6)}$$

This is, of course, the continuity equation-the precise mathematical statement of local conservation of charge. It can be derived from Maxwell's equations also- conservation of charge is not an independent assumption, but a consequence of the laws of electrodynamics.

Problem: Derive the continuity equation from Maxwell equations.
Soluntion: Fourth Maxwell equation is:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{7}$$

Taking divergence of the above equation, we get

$$\nabla.\nabla \times \mathbf{B} = \mu_0 \nabla.\mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \nabla.\mathbf{E}}{\partial t} \tag{8}$$

Since the divergence of curl of any vector is zero, the LHS of the above equation is zero,

$$0 = \mu_0 \nabla.\mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \nabla.\mathbf{E}}{\partial t} \tag{9}$$

Upon simplification we get

$$\nabla.\mathbf{J} + \epsilon_0 \frac{\partial \nabla.\mathbf{E}}{\partial t} = 0 \tag{10}$$

From the first Maxwelll equation, we have

$$\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{11}$$

Substituting above in the equation (10), we obtain

$$\nabla.\mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{12}$$

which is nothing but the continuity equation.

2 Poynting's theorem

The work necessary to assemble a static charge distribution (against the Coulomb repulsion of like charges) is

$$W_e = \frac{1}{2}\int \epsilon_0 E^2 d\tau \tag{13}$$

where $\mathbf{E}$ is the resulting electric field. Likewise, the work required to get currents going (against the back emf) is

$$W_m = \frac{1}{2\mu_0}\int B^2 d\tau \tag{14}$$

where $\mathbf{B}$ is the resulting magnetic field. This suggests that the total energy stored in electromagnetic fields is

$$U_{em} = \int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) d\tau. \tag{15}$$

Let us derive the above equation more generally, now, in the context of the energy conservation law for electrodynamics.

Suppose we have some charge and current configuration which, at time t, produces fields $\mathbf{E}$ and $\mathbf{B}$. In the next instant, dt, the charges move around a bit. *Question*: How much work, dW, is done by the electromagnetic forces acting on these charges in the interval dt) According to the Lorentz force law, the work done on a charge q is

$$\mathbf{F}.\mathbf{dl} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}).\mathbf{v}dt = q\mathbf{E}.\mathbf{v}dt \tag{16}$$

since $\mathbf{v}\times\mathbf{B}).\mathbf{v} = 0$.

Now $q = \rho d\tau$ and $\rho\mathbf{v} = \mathbf{J}$, so the rate at which work is done on all the charges in a volume V is

$$\frac{dW}{dt} = \int_V (\mathbf{E}.\mathbf{J})d\tau. \tag{17}$$

Evidently $\mathbf{E}.\mathbf{J}$ is the work done per unit time, per unit volume-which is to say, the power delivered per unit volume. We can express this quantity in terms of the fields alone. using the Ampere-Maxwell law to eliminate $\mathbf{J}$:

$$\mathbf{E}.\mathbf{J} = \frac{1}{\mu_0}\mathbf{E}.\nabla\times\mathbf{B} - \epsilon_0\mathbf{E}.\frac{\partial\mathbf{E}}{\partial t} \tag{18}$$

From product rule 6,

$$\nabla.(\mathbf{E}\times\mathbf{B}) = \mathbf{B}.\nabla\times\mathbf{E} - \mathbf{E}.\nabla\times\mathbf{B}. \tag{19}$$

Invoking Faraday's law (third Maxwell equation),

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{20}$$

it follows that

$$\mathbf{E}.\nabla \times \mathbf{B} = -\mathbf{B}.\frac{\partial \mathbf{B}}{\partial t} - \nabla.(\mathbf{E} \times \mathbf{B}) \tag{21}$$

Meanwhile,

$$\mathbf{B}.\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial B^2}{\partial t} \tag{22}$$

and

$$\mathbf{E}.\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t} \tag{23}$$

Substituting equation21) in equation (18), we obtain

$$\mathbf{E.J} = -\frac{1}{2\mu_0}\frac{\partial B^2}{\partial t} - \frac{1}{\mu_0}\nabla.(\mathbf{E} \times \mathbf{B}) - \frac{1}{2}\epsilon_0\frac{\partial E^2}{\partial t} \tag{24}$$

Or,

$$\mathbf{E.J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) - \frac{1}{\mu_0}\nabla.(\mathbf{E} \times \mathbf{B}) \tag{25}$$

Putting this into Equation (17),

$$\frac{dW}{dt} = -\frac{1}{2}\int_V \frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) d\tau - \frac{1}{\mu_0}\int_V \nabla.(\mathbf{E} \times \mathbf{B}) d\tau. \tag{26}$$

applying the divergence theorem to the second tenn, we have

$$\frac{dW}{dt} = -\frac{1}{2}\int_V \frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) d\tau - \frac{1}{\mu_0}\oint_S (\mathbf{E} \times \mathbf{B}).\mathbf{da}. \tag{27}$$

where S is the surface bounding V. This is Poynting's theorem; it is the "work-energy theorem" of electrodynamics. The first integral on the right is the total energy stored in the fields, U_{em} (Equation (15)). The second term evidently represents the rate at which energy is carried out of V, across its boundary surface, by the electromagnetic fields. Poynting's theorem says, then, that the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface.

The energy per unit time, per unit area, transported by the fields is called the **Poynting vector**:

$$\boxed{\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}). \qquad (28)}$$

Specifically, $\mathbf{S}.\mathbf{da}$ is the energy per unit time crossing the infinitesimal surface $d\mathbf{a}$-the energy flux,(so $\mathbf{S}$ is the **energy flux density**).

Let us express Poynting's theorem more compactly

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \mathbf{S}.\mathbf{da}. \tag{29}$$

Of course, the work W done on the charges will increase their mechanical energy (kinetic, potential, or whatever). If we let u_{mech} denote the mechanical energy density, so that

$$\frac{dW}{dt} = \frac{d}{dt}\int_V u_{mech} d\tau, \tag{30}$$

and use u_{em} for the energy density of the fields,

$$\boxed{u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right).} \tag{31}$$

Then

$$\frac{dW}{dt} = -\frac{1}{2}\frac{dU_{em}}{dt} - \oint_S \mathbf{S}.\mathbf{da}. \tag{32}$$

becomes

$$\frac{d}{dt}\int_V u_{mech} d\tau = -\frac{dU_{em}}{dt} - \oint_S \mathbf{S}.\mathbf{da}. \tag{33}$$

Or,

$$\frac{d}{dt}\int_V (u_{mech} + u_{em})\, d\tau = -\oint_S \mathbf{S}.\mathbf{da} = -\int_V \nabla.\mathbf{S} d\tau. \tag{34}$$

and hence

$$\boxed{\frac{\partial}{\partial t}(u_{mech} + u_{em}) = -\nabla.\mathbf{S}} \tag{35}$$

This is the differential version of Poynting's theorem. Compare it with the continuity equation, expressing conservation of charge

$$\boxed{\frac{\partial \rho(\mathbf{r}, t)}{\partial t} = -\nabla.\mathbf{J}} \tag{36}$$

the charge density is replaced by the energy density (mechanical plus electromagnetic), and the current density is replaced by the Poynting vector. The latter represents the flow of energy in exactly the same way that $\mathbf{J}$ describes the flow of charge.