Orthogonal Matrix and its Properties

(For B.Sc./B.A. Part-II, Hons. Course of Mathematics)

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Orthogonal Matrix : Definition

A square matrix A is said to be orthogonal if $A'A = I$, where A' denotes the transpose of A and I is a unit matrix of the same order as A.

Properties of an Orthogonal Matrix

I. Every orthogonal matrix is non-singular.

Proof : Let A be an orthogonal matrix. Then, by definition,

$A'A = I$, where A' is the transpose of A and I is a unit matrix of the same order as A.

This $\Rightarrow |A'A| = |I|$

$\Rightarrow |A'|\,|A| = 1$ $\quad$ $\left(\text{since } |A'A| = |A'|\,|A| \text{ and } |I| = 1\right)$

$\Rightarrow |A|\,|A| = 1$ $\quad$ $\left(\text{since } |A'| = |A|\right)$

$\Rightarrow |A|^2 = 1$

$\Rightarrow |A| = \pm 1$

$\Rightarrow |A| \neq 0$

$\Rightarrow A$ is non-singular.

II. I is an orthogonal matrix.

Proof : Let I be a unit matrix. Then, we have

$I'I = I.I$ $\quad$ $\left(\text{since } I' = I\right)$

$= I.$

Therefore I is orthogonal.

III. The product of two orthogonal matrices of the same order is an orthogonal matrix of the same order.

Proof : Let A and B be two orthogonal matrices, each of order n.

To prove : AB is an orthogonal matrix of order n.

Since A and B are orthogonal matrices, therefore, by definition,

$A'A = I$ ... (1)

and $B'B = I$ ... (2)

where A' and B' denote the transpose of A and B respectively and I is the unit matrix of order n.

Now,

$(AB)'(AB) = (B'A')(AB)$ (by reversal rule for transpose of matrices)

$= B'(A'A)B$ (using associative law for multiplication of matrices)

$= B'IB$ (using equation (1))

$= B'B$ $\left(\text{since } IB = B\right)$

$= I$ (using equation (2))

Therefore, AB is an orthogonal matrix of order n.

IV. The transpose of an orthogonal matrix is orthogonal.

Proof : Let A be an orthogonal matrix and let A' denote the transpose of A .

To prove : A' is orthogonal.

Since A is orthogonal, therefore, by definition of an orthogonal matrix,

$$A' A = I \qquad \text{... (1)}$$

where I is a unit matrix of the same order as A.

Equation (1) $\Rightarrow (A' A) A^{-1} = IA^{-1}$

(since A is orthogonal, therefore A is non-singular and consequently, A^{-1} exists)

$\Rightarrow A'(A A^{-1}) = A^{-1}$

(using associative law for multiplication of matrices and $IA^{-1} = A^{-1}$)

$\Rightarrow A' I = A^{-1}$ $\quad (\because\ A A^{-1} = I)$

$\Rightarrow A' = A^{-1}$ $\quad (\because\ A' I = A')$

$\Rightarrow AA' = AA^{-1}$

$\Rightarrow AA' = I$ $\quad (\because\ AA^{-1} = I)$

$\Rightarrow (A')' A' = I$ $\quad \left(\because\ (A')' = A\right)$

$\Rightarrow A'$ is orthogonal.

V. The inverse of an orthogonal matrix is orthogonal.

Proof : Let A be an orthogonal matrix .

To prove : A^{-1} is orthogonal.

Since A is orthogonal, therefore, by definition of an orthogonal matrix,

$$A' A = I \qquad \text{... (1)}$$

where A' denotes the transpose of A and I is a unit matrix of the same order as A.

Also, by definition of inverse of a matrix, we have

$$AA^{-1} = A^{-1}A = I \qquad \text{... (2)}$$

Equation (1) $\Rightarrow (A' A) A^{-1} = IA^{-1}$

$\Rightarrow A'(A A^{-1}) = A^{-1}$

(using associative law for multiplication of matrices and $IA^{-1} = A^{-1}$)

$\Rightarrow A' I = A^{-1}$ $\quad$ (using equation (2))

$\Rightarrow A' = A^{-1}$ $\quad (\because\ A' I = A')$

$\Rightarrow (A')^{-1} = (A^{-1})^{-1}$

$\Rightarrow (A^{-1})' = A$ $\quad \left(\because\ (A')^{-1} = (A^{-1})' \text{ and } (A^{-1})^{-1} = A\right)$

$\Rightarrow (A^{-1})' A^{-1} = AA^{-1}$

$\Rightarrow (A^{-1})' A^{-1} = I$ $\quad$ (using equation (2))

$\Rightarrow A^{-1}$ is orthogonal.

Corollary : For an orthogonal matrix $A,$ we have $A^{-1} = A'$.

Example 1 : Prove that the matrix

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

is orthogonal.

Solution : Let $A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.

Then transpose of A is

$$A' = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Therefore, $A'A = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$= \frac{1}{4}\times 4\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= I.$$

Hence A is orthogonal.

Example 2 : Determine l, m, n so that the matrix

$$A = \begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}$$

may be orthogonal. Hence find A^{-1}.

Solution : First Part

We have to prove that the given matrix A is orthogonal.

A matrix A is said to be orthogonal if

$$A'A = I \qquad \text{... (1)}$$

where A' denotes the transpose of A and I is a unit matrix of the same order as A.

The given matrix is

$$A = \begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}.$$

$$\Rightarrow A' = \begin{bmatrix} 0 & l & l \\ 2m & m & -m \\ n & -n & n \end{bmatrix}$$

$$\Rightarrow A'A = \begin{bmatrix} 0 & l & l \\ 2m & m & -m \\ n & -n & n \end{bmatrix} \begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix} = \begin{bmatrix} 2l^2 & 0 & 0 \\ 0 & 6m^2 & 0 \\ 0 & 0 & 3n^2 \end{bmatrix} \qquad \text{... (2)}$$

Equations (1) and (2) $\Rightarrow \begin{bmatrix} 2l^2 & 0 & 0 \\ 0 & 6m^2 & 0 \\ 0 & 0 & 3n^2 \end{bmatrix} = I$

$$\Rightarrow \begin{bmatrix} 2l^2 & 0 & 0 \\ 0 & 6m^2 & 0 \\ 0 & 0 & 3n^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 2l^2 = 1,\ 6m^2 = 1,\ 3n^2 = 1$$

$$\Rightarrow l^2 = \frac{1}{2},\ m^2 = \frac{1}{6},\ n^2 = \frac{1}{3}$$

$$\Rightarrow l = \pm\frac{1}{\sqrt{2}},\ m = \pm\frac{1}{\sqrt{6}},\ n = \pm\frac{1}{\sqrt{3}}.$$

Second Part

We have to find A^{-1}.

From equation (1), we have

$$A'A = I$$

This $\Rightarrow (A'A)A^{-1} = IA^{-1}$

$\Rightarrow A'(A A^{-1}) = A^{-1}$ (using associative law for multiplication of matrices and $IA^{-1} = A^{-1}$)

$\Rightarrow A'I = A^{-1}$ $(\because\ A A^{-1} = I)$

$\Rightarrow A' = A^{-1}$ $(\because\ A' I = A')$

$$\Rightarrow A^{-1} = \begin{bmatrix} 0 & l & l \\ 2m & m & -m \\ n & -n & n \end{bmatrix}.$$

Exercises

1. Prove that the matrix

$$A = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

is orthogonal. Hence find A^{-1}.

2. Determine a, b, c so that the matrix

$$A = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$$

may be orthogonal.

Answers

1. $A^{-1} = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$

2. $a = \frac{2}{3},\ b = -\frac{2}{3},\ c = \frac{1}{3}$ or $a = -\frac{2}{3},\ b = \frac{2}{3},\ c = -\frac{1}{3}$