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Program for B.Sc (Hons) Part-1

Topic: - Maximal and Minimal elements of a partially ordered set.

Partially ordered set - A set with a partial order relation is called partially ordered set. Ex- Let IN= {1, 2, 3, ...} be the set of natural numbers. Again let the relation ≤ be defined on IN is such a way that a ≤b if a divides b. Now (N, ≤) is a partially ordered set.

Totally ordered set – A set with a total order relation is called a liner ordered set or totally order set or a chain.

Ex: - Let IR = Set of real numbers. Also let the relation on IR is defined by '≤' where '≤' has its usual meaning. Now it is very clear that (IR, ≤) is a totally ordered set. The main reason of (IR, S) to be a totally order set is that for any a, b ϵ IR, either a ≤ b or b ≤a.

Note:- By the definition, every totally ordered set is partially ordered set .But the converse is not true. That is, a partially ordered set need not be totally ordered set.

Ex: - Let the set of positive integers

IN = {1, 2,3, ...}. Also let the relation '≤' be defined on IN such way that a ≤ b if a divides b. Here (IN, ≤) is a partially order set but not a totally ordered set. Because for all a, b ∈ IN, either a ≤b or b ≤ a does not hold true.

(The least and greatest elements of a partially ordered set) :- Consider (S , ≤) be a partially ordered set and suppose x , y ϵ S .

If x ≤y and x≠ y, then x is called strictly smaller than y and we write x <y.

i. An element a ϵ S is called the least or the first element of s if a ≤ x , for all x ϵ S.
ii. An element a ϵ S is called greatest or the last element of S if x ≤ a ∀ x ϵ S.

For example:-

Let IN = set of natural number S with their natural order 1,2,3,4 … has the first elements 1 but no last elements

It the set is partially ordered as …………. , 4,3,2,1 then it has no first elements but has the last element 1.

Again by ordering 1,3, 5, …….,6,4,2 has the fist elements 1 and the last element to where as …….5, 3,4,6….. has neither a first element nor a last element.

Note – The least (or the greatest) element of a partially ordered set (S, ≤), if it exists, is unique.

(Maximal and Minimal elements of a partially ordered set) :-

i. Let (S, ≤) be a partially ordered set. Now, an element a ϵ S is called the minimal element of S if x ≤a ⇒ a=x where x ϵ S. Therefore an element a ϵ S is a minimal element of S if there is no element in S which strictly preceds a.

ii. Let (S, ≤) be partially ordered set. Now an element m ϵ S is called the maximal element of S if m ≤ x ⇒ m = x where x ϵ S. Thus an element m ϵ S is a maximal element in S if there is no element in S which strictly following dominates m.

Example –

IN=Set of natural number. Let '≤' is the usual order relation on IN. Now, we get minimal element.

Here (IN, ≤) is a partially ordered set. Clearly 1 is the least element of IN, since 1≤m, ⊬ m ϵ IN and there is no greatest or last element of N. Again , 1 is also the only minimal element , since if x ϵIN, then x ≤1⇒ x =1.

Important facts about Maximal and Minimal elements of a partially ordered set:-

i. A partially ordered set may have no maximal element, or may have one or more maximal elements.

For example, IR= the set of real numbers with the natural ordering is a partially ordered set which is also totally ordered but it has no maximal element and no minimal element.

ii. It is very clear that if a is the first element of S, then a is a minimal element of S and the only one.

And if S contains the last element b, then b is a maximal element and the only one. Also, a totally ordered set can contain at most one maximal element which would be the last element. Similarly it can contain at most one minimal element which would be the first element.

iii. Every finite partially ordered set has at least one maximal element and at least one minimal element.

Some examples on Maximal and Minimal element of a partially ordered set:-

Ex: 1 . Let the set S = { 1, 2, 3,,4,12} and also let '≤' be a relation defined on S such that a ≤ b if a divides b. Find the maximal elements of S.

Solution :- Here (S,≤) is a partially ordered set. Clearly 1 divides each of the number 1, 2, 3, 4, 12 and so 1 ≤ x ⊬ x ϵ S.

Therefore 1 is the least element of s.

Again x ≤ 12 ⊬ x ϵ S i.e. Each elements of S divides 12. Therefore 12 is the greatest element of S.

Here, also 1 is the only minimal element, since if x ϵ S, then x ≤ 1 i.e. x divides 1⇒ x= 1. Also 12 is the only maximal element of S.

Ex :-2 The totally ordered set A={ x: o<x<1} with ordering '≤' has no first and no last elements. Also , it has no maximal element and no minimal element.

Ex:-3 Let P(S)= the collection of all subsets of a non-empty s ordered by the set inclusion ⊑. Clearly P(S, ⊑) is a partially ordered set. For any elements A, B ∈P (S), let A ≤B means that A ⊑ B. Then the empty set ∅ is the least element and S is the greatest element of (P (S), ⊑) for ∅ ⊑ A⊬ A ϵ P (S) and D⊆S ⊬ DϵP (S).