Solved Examples of Gradient

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Let us first define a vector differential operator ∇ (pronounced as dell). ∇ is defined as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \tag{1}$$

This vector operator possesses property analogous to those of all ordinary vectors. It is useful in defining three quantities which arise in practical applications and are known as the gradient (∇), divergence ($\nabla.$) and curl ($\nabla\times$).

1 The Gradient of Scalar Fields

Let $\Phi(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space i.e. Φ defines a differentiable scalar field. Then the gradient of Φ, written as grad Φ or $\nabla\Phi$, is defined as

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k} \tag{2}$$

2 Examples of Gradient

1. If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ (or grad ϕ) at the point $(1,-2,-1)$.
   Sol.:

$$\begin{aligned}
\nabla\phi &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(3x^2y - y^3z^2) \\
&= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2) \\
&= 6xy\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2y^3z\hat{k}
\end{aligned}$$

   Let us substitute $(x,y,z) = (1,-2,-1)$ in the above equation to find grad ϕ at the point $(1,-2,-1)$

$$\begin{aligned}
\nabla\phi &= 6(1)(-2)\hat{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\hat{j} - 2(-2)^3(-1)\hat{k} \\
&= -12\hat{i} - 9\hat{j} - 16\hat{k}
\end{aligned}$$

2. Prove
   (a) $\nabla(F+G) = \nabla F + \nabla G$,
   (b) $\nabla(FG) = F\nabla G + G\nabla F$

   where F and G are differentiable scalar functions of x,y and z.

   (a) **Sol:**

$$\begin{aligned}
\nabla(F+G) &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(F+G) \\
&= \hat{i}\frac{\partial}{\partial x}(F+G) + \hat{j}\frac{\partial}{\partial y}(F+G) + \hat{k}\frac{\partial}{\partial z}(F+G) \\
&= \hat{i}\frac{\partial}{\partial x}F + \hat{i}\frac{\partial}{\partial x}G + \hat{j}\frac{\partial}{\partial y}F + \hat{j}\frac{\partial}{\partial y}G + \hat{k}\frac{\partial}{\partial z}F + \hat{k}\frac{\partial}{\partial z}G \\
&= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)F + \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)G \\
&= \nabla F + \nabla G
\end{aligned}$$

Proved

(b) **Sol:**

$$\begin{aligned}
\nabla(FG) &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(FG) \\
&= \hat{i}\frac{\partial}{\partial x}(FG) + \hat{j}\frac{\partial}{\partial y}(FG) + \hat{k}\frac{\partial}{\partial z}(FG) \\
&= G\hat{i}\frac{\partial}{\partial x}F + F\hat{i}\frac{\partial}{\partial x}G + G\hat{j}\frac{\partial}{\partial y}F + F\hat{j}\frac{\partial}{\partial y}G + G\hat{k}\frac{\partial}{\partial z}F + F\hat{k}\frac{\partial}{\partial z}G \\
&= G\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)F + F\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)G \\
&= G\nabla F + F\nabla G
\end{aligned}$$

Proved

3. Find $\nabla\phi$ if

   (a) $\phi = \ln|\mathbf{r}|$,
   (b) $\phi = \frac{1}{r}$.

   Sol:

   (a) Position vector

$$\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

   Then

$$|\mathbf{r}| = \sqrt{x^2+y^2+z^2}$$

   and

$$\phi = \ln|\mathbf{r}| = r = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2).$$

$$\begin{aligned}
\nabla\phi &= \frac{1}{2}\nabla\ln(x^2+y^2+z^2). \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\ln(x^2+y^2+z^2) \\
&= \frac{1}{2}\left(\hat{i}\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) + \hat{j}\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + \hat{k}\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\right) \\
&= \frac{1}{2}\left(\hat{i}\frac{2x}{x^2+y^2+z^2} + \hat{j}\frac{2y}{x^2+y^2+z^2} + \hat{k}\frac{2z}{x^2+y^2+z^2}\right) \\
&= \left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}\right) \\
&= \frac{\mathbf{r}}{r^2}
\end{aligned}$$

(b)

$$
\begin{aligned}
\nabla\phi &= \nabla\frac{1}{r} \\
&= \nabla\frac{1}{\sqrt{x^2+y^2+z^2}} \\
&= \nabla(x^2+y^2+z^2)^{-1/2} \\
&= \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)^{-1/2} \\
&= \left(\hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}+\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}+\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}\right) \\
&= \left(\hat{i}\frac{-2x}{2(x^2+y^2+z^2)^{3/2}}+\hat{j}\frac{-2y}{2(x^2+y^2+z^2)^{3/2}}+\hat{k}\frac{-2z}{2(x^2+y^2+z^2)^{3/2}}\right) \\
&= \left(-\frac{x\hat{i}-y\hat{j}-z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right) \\
&= \frac{-\mathbf{r}}{r^3}
\end{aligned}
$$

4. Show that $\nabla r^n = nr^{n-2}\mathbf{r}$.
   Sol:

$$
\begin{aligned}
\nabla r^n &= \nabla(\sqrt{x^2+y^2+z^2})^n \\
&= \nabla(x^2+y^2+z^2)^{n/2} \\
&= \nabla(x^2+y^2+z^2)^{-1/2} \\
&= \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)^{n/2} \\
&= \left(\hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}+\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2}+\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}\right) \\
&= \left(\hat{i}\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}2x+\hat{j}\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}2y+\hat{k}\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}2z\right) \\
&= n(x^2+y^2+z^2)^{n/2-1}\left(x\hat{i}+y\hat{j}+z\hat{k}\right) \\
&= n(r^2)^{n/2-1}\mathbf{r} \\
&= nr^{n-2}\mathbf{r}
\end{aligned}
$$

Proved.

3 Homework:

1. If $\phi = 2xz^4 - x^2y$, find $\nabla\phi$ and $|\nabla\phi|$ at the point $(2, -2, -1)$.
2. If $\mathbf{A} = 2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k}$ and $\phi = 2z - x^3y$, find $\mathbf{A}.\nabla\phi$ and $\mathbf{A} \times \nabla\phi$ at the point $(1, -1, 1)$.
3. If $F = x^2z + e^{y/x}$ and $G = 2z^2y - xy^2$, find
   (a) $\nabla(F + G)$ and
   (b) $\nabla(FG)$

   at the point $(1, 0, -2)$.
4. Find ∇r^3.
5. $\nabla(fr) = \frac{f'(r)\mathbf{r}}{r}$.
6. Evaluate $\nabla(3r^2 - 4\sqrt{r} + \frac{6}{r^{1/3}})$.
7. If $\nabla U = 2r^4r$, find U.
8. Find $\phi(r)$ such that $\nabla\phi = \frac{\mathbf{r}}{r^5}$ and $\phi(1) = 0$.
9. Find $\nabla\psi$ where $\psi = (x^2 + y^2 + z^2)e^{-\sqrt{x^2+y^2+z^2}}$.
10. If $\nabla\phi = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ and $\phi(1, -2, 2) = 4$, find $\phi(x, y, z)$.
11. If $\mathbf{A}$ is a constant vector, prove $\nabla(\mathbf{r}.\mathbf{A})\mathbf{A}$.
12. If U is a differentiable function of (x, y, z), prove $\nabla U.d\mathbf{r} = dU$.