Elementary Transformations and Elementary Matrices

(For B.Sc./B.A. Part-II, Hons. Course of Mathematics)

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Contents

1. Elementary Transformations of a Matrix

Definition : Elementary Transformations of a matrix means some operations performed on rows or columns of the matrix to transform it into a different form so that the calculations become simpler.

Elementary transformations performed on rows of a matrix are known as elementary row transformations and elementary transformations performed on columns of a matrix are known as elementary column transformations.

There are six elementary transformations of a matrix, three due to rows and three due to columns :

(i) Interchange of any two rows (or columns) of a matrix
(ii) The multiplication of all the elements of any row (or columns) of a matrix by a non zero number
(iii) The addition to all the elements of any row (or column) of a matrix, the corresponding elements of any other row (or column) multiplied by some number

Notations for Elementary Transformations

The following notations are used to denote the elementary transformations :

(i) R_{ij} : It stands for the interchange of the $i\text{th}$ and $j\text{th}$ rows of a matrix.

(ii) $R_i(c)$: It stands for multiplication of the $i\text{th}$ row of a matrix by a number $c \neq 0$.

(iii) $R_{ij}(k)$: It stands for addition to the $i\text{th}$ row of a matrix, the product of the $j\text{th}$ row by a number k.

The corresponding column transformations are denoted by C in place of R, i.e., by C_{ij}, $C_i(c)$ and $C_{ij}(k)$ respectively.

Equivalent Matrices

If a matrix B can be obtained from a matrix A by performing a finite number of elementary row (or column) transformations, then B is said to be equivalent to A and it is written as $B \sim A$.

Note : It can be proved that equivalence of matrices is an equivalence relation.

2. Elementary Matrices

Definition : A matrix obtained from a unit matrix by subjecting it to any of the elementary transformations is called an elementary matrix.

There are four types of elementary matrices :

(i) E_{ij} : It stands for a matrix obtained from a unit matrix by interchanging its $i\text{th}$ and $j\text{th}$ rows (or columns).

(ii) $E_i(c)$: It stands for a matrix obtained from a unit matrix by multiplying its $i\text{th}$ row (or column) by a number $c \neq 0$.

(iii) $E_{ij}(k)$: It stands for a matrix obtained from a unit matrix by addition to its ith row (or column), the product of its jth row (or column) by a number k.

(iv) $E'_{ij}(k)$: It stands for the transpose of $E_{ij}(k)$.

Theorem : The elementary matrices are non-singular.

Proof : There are four types of elementary matrices : $E_{ij}, E_i(c), E_{ij}(k)$ and $E'_{ij}(k)$.

We have to prove that each of these is non-singular.

E_{ij} is non-singular

We know that if any two rows or columns of a determinant are interchanged, then its value remains the same but the sign is changed. Therefore,

$$|E_{ij}| = -|I|, \quad \text{where } I \text{ denotes the unit matrix}$$
$$= -1 \qquad (\because |I| = 1)$$
$$\neq 0$$

This $\Rightarrow E_{ij}$ is non-singular.

$E_i(c)$ is non-singular

We know that if each element of any row or column of a determinant is multiplied by a non-zero number c, then the determinant is multiplied by c. Therefore,

$$|E_i(c)| = c|I|$$
$$= c.1 \qquad (\because |I| = 1)$$
$$= c$$
$$\neq 0$$

This $\Rightarrow E_i(c)$ is non-singular.

$E_{ij}(k)$ is non-singular

We know that a determinant remains unaltered in value by adding to all the elements of any row or column, a constant multiple of the corresponding elements of any other row or column. Therefore,

$$|E_{ij}(k)| = |I|$$
$$= 1 \qquad (\because |I| = 1)$$
$$\neq 0$$

This $\Rightarrow E_{ij}(k)$ is non-singular.

$E'_{ij}(k)$ is non-singular

Since a matrix A and its transpose A' have the same determinant, therefore

$$|E'_{ij}(k)| = |E_{ij}(k)|$$
$$= 1 \qquad (\because |E_{ij}(k)| = 1, \text{as proved earlier})$$
$$\neq 0$$

This $\Rightarrow E'_{ij}(k)$ is non-singular.

3. Elementary Transformations and Elementary Matrices

Theorem 1.

(a) Every elementary row transformation on the product AB of two matrices A and B is equivalent to the same elementary row transformation on the pre-factor A of AB.

(b) Every elementary column transformation on the product AB of two matrices A and B is equivalent to the same elementary column transformation on the post-factor B of AB.

Proof : Let A and B be two matrices of orders $m\times n$ and $n\times p$ respectively so that the product AB is defined.

Let the rows of A be denoted by $R_1, R_2, R_3, \ldots, R_m$ and let the columns of B be denoted by $C_1, C_2, C_3, \ldots, C_p$. Then, we have

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \ldots \\ \ldots \\ R_m \end{bmatrix} \quad \text{and } B = \begin{bmatrix} C_1 & C_2 & \ldots & C_p \end{bmatrix}.$$

Therefore,

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & \ldots & R_1C_p \\ R_2C_1 & R_2C_2 & \ldots & R_2C_p \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ R_mC_1 & R_mC_2 & \ldots & R_mC_p \end{bmatrix}$$

This representation of the matrix AB shows that if the rows $R_1, R_2, R_3, \ldots, R_m$ of A be subjected to any elementary transformation, then the rows of AB are also subjected to the same elementary transformation. This proves (a).

Similarly, the product AB indicates that if the columns $C_1, C_2, C_3, \ldots, C_p$. of B be subjected to any elementary transformation, then the columns of AB are also subjected to the same elementary transformation. This proves (b).

Theorem 2.

(a) Every elementary row transformation of a matrix can be brought about by pre-multiplication with the corresponding elementary matrix.

(b) Every elementary column transformation of a matrix can be brought about by post-multiplication with the corresponding elementary matrix.

Proof : (a) Let A be any matrix and I, the unit matrix. Then, we can write

$$A = IA \qquad \ldots(1)$$

Now, let us apply some elementary row transformation λ on A and let the matrix obtained from A by applying this elementary row transformation λ be denoted by $(A)_\lambda$. Then, it follows from equation (1) that

$$(A)_\lambda = (IA)_\lambda$$

This $\Rightarrow (A)_\lambda = (I)_\lambda A$

[Since every elementary row transformation on the product AB of two matrices A and B is equivalent to the same elementary row transformation on the pre-factor A of AB.]

$\Rightarrow (A)_\lambda = EA,$

where E denotes the elementary matrix corresponding to the elementary row transformation λ. This proves (a).

(b) If A be any matrix and I, the unit matrix, then, we can write

$$A = AI \qquad \text{...(2)}$$

Now, let us apply some elementary column transformation μ on A and let the matrix obtained from A by applying this elementary column transformation μ be denoted by $(A)_\mu$. Then, it follows from equation (2) that

$$(A)_\mu = (AI)_\mu$$

This $\Rightarrow (A)_\mu = A(I)_\mu$

[Since every elementary column transformation on the product AB of two matrices A and B is equivalent to the same elementary column transformation on the post-factor B of AB.]

$\Rightarrow (A)_\mu = AE,$

where E denotes the elementary matrix corresponding to the elementary column transformation μ. This proves (b).

Theorem 3. The inverse of a non-singular matrix can be obtained by elementary transformations.

Proof : Let A be a non-singular matrix.

We have to prove that A^{-1} can be obtained by elementary transformations.

Let us apply some elementary row transformations on the matrix A until A is reduced to the unit matrix I. This is equivalent to pre-multiplying the matrix A by the corresponding elementary matrices $E_1, E_2, \ldots, E_k$ until A is reduced to the unit matrix I. Therefore, we have

$$(E_1 \, E_2 \ldots E_k)A = I$$

This $\Rightarrow (E_1 \, E_2 \ldots E_k)AA^{-1} = IA^{-1}$

$\Rightarrow (E_1 \, E_2 \ldots E_k)I = A^{-1} \qquad \left[\because AA^{-1} = I \quad \text{and} \quad IA^{-1} = A^{-1}\right]$

$\Rightarrow$ If the matrix A is reduced to the unit matrix I by some elementary row transformations, the unit matrix I is reduced to A^{-1} by same elementary row transformations.

Similarly, if we have to find A^{-1} using elementary column transformations, then we apply some elementary column transformations on the matrix A until A is reduced to the unit matrix I. This is equivalent to post-multiplying the matrix A by the corresponding elementary matrices $E_1, E_2, \ldots, E_k$ until A is reduced to I. Therefore, we have

$$A(E_1 \, E_2 \ldots E_k) = I$$

This $\Rightarrow A^{-1}A(E_1 \, E_2 \ldots E_k) = A^{-1}I$

$\Rightarrow I(E_1 \, E_2 \ldots E_k) = A^{-1} \qquad \left[\because A^{-1}A = I \quad \text{and} \quad A^{-1}I = A^{-1}\right]$

$\Rightarrow$ If the matrix A is reduced to the unit matrix I by some elementary column transformations, the unit matrix I is reduced to A^{-1} by same elementary column transformations.

Example : Compute the inverse of the matrix

$$A=\begin{bmatrix}-1 & -3 & 3 & -1\\ 1 & 1 & -1 & 0\\ 2 & -5 & 2 & -3\\ -1 & 1 & 0 & 1\end{bmatrix}$$

Solution : A matrix B is the inverse of a matrix A, if

$BA = I$, where I is the unit matrix of the same order as A.

For the given matrix A, we can write

$$A = IA, \qquad \text{...(1)}$$

where I is the unit matrix of order 4.

Now, we apply elementary row transformations on the matrix A on the LHS of equation (1) in order to transform it to the unit matrix I.

From equation (1), we have

$$\begin{bmatrix}-1 & -3 & 3 & -1\\ 1 & 1 & -1 & 0\\ 2 & -5 & 2 & -3\\ -1 & 1 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}A$$

This $\Rightarrow \begin{bmatrix}1 & 3 & -3 & 1\\ 1 & 1 & -1 & 0\\ 2 & -5 & 2 & -3\\ -1 & 1 & 0 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}A \qquad (\text{Replacing } R_1 \text{ by} - R_1)$

$$\Rightarrow \begin{bmatrix}1 & 3 & -3 & 1\\ 0 & -2 & 2 & -1\\ 0 & -11 & 8 & -5\\ 0 & 4 & -3 & 2\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 2 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix}A$$

$(\text{Replacing } R_2 \text{ by } R_2 - R_1,\ R_3 \text{ by } R_3 - 2R_1 \text{ and } R_4 \text{ by } R_4 + R_1)$

$$\Rightarrow \begin{bmatrix}1 & 3 & -3 & 1\\ 0 & 1 & -1 & 1/2\\ 0 & -11 & 8 & -5\\ 0 & 4 & -3 & 2\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0 & 0\\ -1/2 & -1/2 & 0 & 0\\ 2 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix}A \qquad \left(\text{Replacing } R_2 \text{ by} -\frac{1}{2}R_2\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & -3 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 \\ -7/2 & -11/2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$\left(\text{Replacing } R_1 \text{ by } R_1 - 3R_2,\ R_3 \text{ by } R_3 + 11R_2 \text{ and } R_4 \text{ by } R_4 - 4R_2\right)$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & -1/6 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 \\ 7/6 & 11/6 & -1/3 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A \qquad \left(\text{Replacing } R_3 \text{ by} -\frac{1}{3} R_3\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & -1/6 \\ 0 & 0 & 0 & 1/6 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 & 0 & 0 \\ 2/3 & 4/3 & -1/3 & 0 \\ 7/6 & 11/6 & -1/3 & 0 \\ -1/6 & 1/6 & 1/3 & 1 \end{bmatrix} A$$

$\left(\text{Replacing } R_2 \text{ by } R_2 + R_3 \text{ and } R_4 \text{ by } R_4 - R_3\right)$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & -1/6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 & 0 & 0 \\ 2/3 & 4/3 & -1/3 & 0 \\ 7/6 & 11/6 & -1/3 & 0 \\ -1 & 1 & 2 & 6 \end{bmatrix} A \qquad \left(\text{Replacing } R_4 \text{ by } 6R_4\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$\left(\text{Replacing } R_1 \text{ by } R_1 + \frac{1}{2} R_4,\ R_2 \text{ by } R_2 - \frac{1}{3} R_4 \text{ and } R_3 \text{ by } R_3 + \frac{1}{6} R_4\right)$

$$\Rightarrow I = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$\Rightarrow I = B\,A,$

where $B = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$ is the inverse of A.

Exercises

Compute the inverse of the following matrices :

1. $\begin{bmatrix} 0 & 2 & 1 & 13 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$

Answers

1. $\begin{bmatrix} 1 & 3 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ 8 & 13 & -16 & 3 \\ -3 & -5 & 6 & -1 \end{bmatrix}$

2. $\begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$