Functions of complex Variable

(B.A/B.Sc, Part-III, Hons.)

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FUNCTIONS OF COMPLEX VARIABLE, ANALYTIC FUNCTION

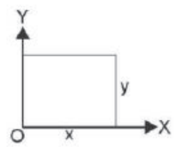
1 INTRODUCTION

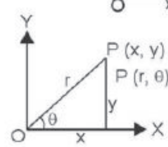
The theory of functions of a complex variable is of atmost importance in solving a large number of problems in the field of engineering and science. Many complicated integrals of real functions are solved with the help of functions of a complex variable.

2 COMPLEX VARIABLE

$x + iy$ is a complex variable and it is denoted by z.

(1) $z = x + iy$. where $i = \sqrt{-1}$ (Cartesian form)





(2) $z = r(\cos\theta + i\sin\theta)$(Polar form)

(3) $z = re^{i\theta}$ (Exponential form)

3 FUNCTIONS OF A COMPLEX VARIABLE

$f(z)$ is a function of a complex variable z and is denoted by w.

$$w = f(z)$$
$$w = u + iv$$

where u and v are the real and imaginary parts of $f(z)$.

4 LIMIT OF A FUNCTION OF A COMPLEX VARIABLE

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0. Then the limit of $f(z)$ as z approaches z_0 is w_0.

$$\lim_{z \to z_0} f(z) = w_0$$

Example. Prove that $\lim\limits_{z \to 1-i} \dfrac{(z^2 + 4z + 3)}{z+1} = 4 - i$

Solution. $\lim\limits_{z \to 1-i} \dfrac{z^2 + 4z + 3}{z+1} = \lim\limits_{z \to 1-i} \dfrac{(z+1)(z+3)}{(z+1)} = \lim\limits_{z \to 1-i} (z+3) = (1-i) + 3 = 4 - i$ **Proved.**

5 CONTINUITY

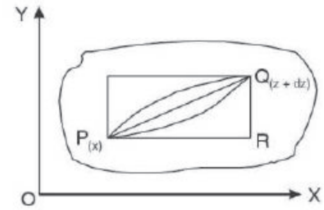
The function $f(z)$ of a complex variable z is said to be continuous at the point z_0 if for any given positive number $\in$, we can find a number δ such that $|f(z) - f(z_0)| < \in$

for all points z of the domain satisfying

$$|z - z_0| < \delta$$

$f(z)$ is said to be continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$



6 DIFFERENTIABILITY

Let $f(z)$ be a single valued function of the variable z, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

provided that the limit exists and is independent of the path along which $\delta z \to 0$.

Let P be a fixed point and Q be a neighbouring point. The point Q may approach P along any straight line or curved path.

Example 1. *Consider the function*

$$f(z) = 4x + y + i(-x + 4y)$$

and discuss $\dfrac{df}{dz}$.

Solution. Here, $f(z) = 4x + y + i(-x + 4y)$

$$= u + iv$$

so $u = 4x + y$

and $v = -x + 4y$

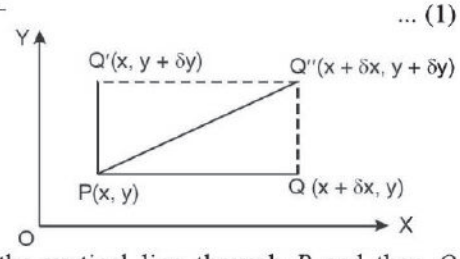
$$f(z+\delta z) = 4(x+\delta x) + (y+\delta y) - i(x+\delta x) + 4i(y+\delta y)$$

$$f(z+\delta z) - f(z) = 4(x+\delta x) + (y+\delta y) - i(x+\delta x) + 4i(y+\delta y) - 4x - y + ix - 4iy$$

$$= 4\delta x + \delta y - i\,\delta x + 4i\delta y$$

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{4\delta x + \delta y - i\,\delta x + 4i\delta y}{\delta x + i\delta y}$$

$$\Rightarrow \qquad \frac{\delta f}{\delta z} = \frac{4\delta x + \delta y - i\delta x + 4i\delta y}{\delta x + i\delta y} \qquad \ldots (1)$$



(*a*) **Along real axis:** If Q is taken on the horizontal line through $P(x, y)$ and Q then approaches P along this line, we shall have $\delta y = 0$ and $\delta z = \delta x$.

$$\frac{\delta f}{\delta z} = \frac{4\delta x - i\delta x}{\delta x} = 4 - i$$

(*b*) **Along imaginary axis:** If Q is taken on the vertical line through P and then Q approaches P along this line, we have

$$z = x + iy = 0 + iy, \ \delta z = i\delta y, \ \delta x = 0.$$

Putting these values in (1), we have

$$\frac{\delta f}{\delta z} = \frac{\delta y + 4i\delta y}{i\delta y} = \frac{1}{i}(1 + 4i) = 4 - i$$

(*c*) **Along a line $y = x$:** If Q is taken on a line $y = x$.

$$z = x + iy = x + ix = (1+i)\,x$$

$$\delta z = (1+i)\,\delta x \quad \text{and} \quad \delta y = \delta x$$

On putting these values in (1), we have

$$\frac{\delta f}{\delta z}=\frac{4\delta x+\delta x-i\delta x+4i\delta x}{\delta x+i\delta x}=\frac{4+1-i+4i}{1+i}=\frac{5+3i}{1+i}=\frac{(5+3i)(1-i)}{(1+i)(1-i)}=4-i$$

In all the three different paths approaching Q from P, we get the same values of $\frac{\delta f}{\delta z}=4-i$.

In such a case, the function is said to be differentiable at the point z in the given region.

Example 2. *If* $f(z)=\begin{cases}\frac{x^3y(y-ix)}{x^6+y^2}, & z\neq 0,\\ 0, & z=0\end{cases}$ *then discuss* $\frac{df}{dz}$ *at* $z = 0$.

Solution. If $z\to 0$ along radius vector $y = mx$

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{z\to 0}\left[\frac{\frac{x^3y(y-ix)}{x^6+y^2}-0}{x+iy}\right]=\lim_{z\to 0}\left[\frac{-ix^3y(x+iy)}{(x^6+y^2)(x+iy)}\right]$$

$$=\lim_{z\to 0}\left[\frac{-ix^3y}{x^6+y^2}\right]=\lim_{z\to 0}\left[\frac{-ix^3(mx)}{x^6+m^2x^2}\right]\qquad [\because\ y=mx]$$

$$=\lim_{x\to 0}\left[\frac{-imx^2}{x^4+m^2}\right]=0$$

But along $y = x^3$

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{z\to 0}\left[\frac{-ix^3y}{x^6+y^2}\right]=\lim_{x\to 0}\frac{-ix^3(x^3)}{x^6+(x^3)^2}=-\frac{i}{2}$$

In different paths we get different values of $\frac{df}{dz}$ *i.e.* 0 and $\frac{-i}{2}$. In such a case, the function is not differentiable at $z = 0$.

Theorem: Continuity is a necessary condition but not sufficient condition for the existence of a finite derivative.

Proof. We have, $f(z_0+\delta z)-f(z_0)=\delta z\left\{\frac{f(z_0+\delta z)-f(z_0)}{\delta z}\right\}$ ... (1)

Taking limit of both sides of (1), as $\delta z\to 0$, we get

$$\operatorname*{Lim}_{\delta z\to 0}\left[f(z_0+\delta z)-f(z_0)\right]=0.f'(z_0)\ \Rightarrow\ \operatorname*{Lim}_{\delta z\to 0}\left[f(z_0+\delta z)-f(z_0)\right]=0$$

$$\Rightarrow\quad \operatorname*{Lim}_{z\to z_0}[f(z)-f(z_0)]=0\quad\Rightarrow\quad \operatorname*{Lim}_{z\to z_0}f(z)=f(z_0)$$

$\Rightarrow$ $f(z)$ is continuous at $z=z_0$. **Proved.**

The converse of the above theorem is not true.

This can be shown by the following example.

Example 3. *Prove that the function* $f(z)=|z|^2$ *is continuous everywhere but no where differentiable except at the origin.*

Solution. Here, $f(z) = |z|^2$.

$\therefore$ But $|z|=\sqrt{(x^2+y^2)}$ $\Rightarrow$ $|z|^2=x^2+y^2$

Since x^2 and y^2 are polynomial so x^2+y^2 is continuous everywhere, therefore, $|z|^2$ is continuous everywhere.

Now, we have $f'(z) = \lim_{\delta z \to 0} \dfrac{f(z+\delta z) - f(z)}{\delta z}$

$$f'(z) = \lim_{\delta z \to 0} \frac{|z+\delta z|^2 - |z|^2}{\delta z} \qquad (z\bar{z} = |z|^2)$$

$$= \lim_{\delta z \to 0} \frac{(z+\delta z)(\bar{z}+\delta \bar{z}) - z\bar{z}}{\delta z} = \lim_{\delta z \to 0} \frac{z\bar{z} + z\delta\bar{z} + \delta z.\bar{z} + \delta z.\delta\bar{z} - z\bar{z}}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{z.\delta\bar{z} + \delta z.\bar{z} + \delta z\delta\bar{z}}{\delta z} = \lim_{\delta z \to 0} \left\{\bar{z} + \delta\bar{z} + z\frac{\delta\bar{z}}{\delta z}\right\} = \lim_{\delta z \to 0} \left\{\bar{z} + z\frac{\delta\bar{z}}{\delta z}\right\} \quad ...(1)$$

[Since, $\delta z \to 0$ so $\delta\bar{z} \to 0$]

Let $\delta z = r(\cos\theta + i\sin\theta)$ and $\delta\bar{z} = r(\cos\theta - i\sin\theta)$

$$\Rightarrow \quad \frac{\delta\bar{z}}{\delta z} = \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta} \qquad \Rightarrow \quad \frac{\delta\bar{z}}{\delta z} = (\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)^{-1}$$

$$\Rightarrow \quad \frac{\delta\bar{z}}{\delta z} = (\cos\theta - i\sin\theta)(\cos\theta - i\sin\theta) \quad \Rightarrow \quad \frac{\delta\bar{z}}{\delta z} = (\cos\theta - i\sin\theta)^2$$

$$\Rightarrow \quad \frac{\delta\bar{z}}{\delta z} = \cos 2\theta - i\sin 2\theta$$

Since $\dfrac{\delta\bar{z}}{\delta z}$ depends on θ. It means for different values of θ, $\dfrac{\delta\bar{z}}{\delta z}$ has different values.

It means $\dfrac{\delta\bar{z}}{\delta z}$ has different values for different z.

Therefore $\lim_{\delta z \to 0} \dfrac{\delta\bar{z}}{\delta z}$ does not tend to a unique limit when $z \neq 0$.

Thus, from (1), it follows that $f'(z)$ is not unique and hence $f(z)$ is not differentiable when $z \neq 0$.

But when $z = 0$ then $f'(z) = 0$ *i.e.*, $f'(0) = 0$ and is unique.

Hence, the function is differentiable at $z = 0$. **Ans.**

7 ANALYTIC FUNCTION

A function $f(z)$ is said to be **analytic** at a point z_0, if f is differentiable not only at z_0 but at every point of some neighbourhood of z_0.

A function $f(z)$ is analytic in a domain if it is **analytic** at every point of the domain.

The point at which the function is not differentiable is called a **singular point** of the function.

An analytic function is also known as "holomorphic", "regular", "monogenic".

Entire Function. A function which is analytic everywhere (for all z in the complex plane) is known as an entire function.

For Example 1. Polynomials rational functions are entire.

2. $|\bar{z}|^2$ is differentiable only at $z = 0$. So it is no where analytic.

Note: (*i*) An entire is always analytic, differentiable and continuous function. But converse is not true.

(*ii*) Analytic function is always differentiable and continuous. But converse is not true.

(*iii*) A differentiable function is always continuous. But converse is not true

8 THE NECESSARY CONDITION FOR $f(z)$ TO BE ANALYTIC

Theorem. *The necessary conditions for a function $f(z) = u + iv$ to be analytic at all the points in a region R are*

(*i*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ (*ii*) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ provided $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ exist.

Proof: Let $f(z)$ be an analytic function in a region R,

$$f(z) = u + iv,$$

where u and v are the functions of x and y.

Let δu and δv be the increments of u and v respectively corresponding to increments δx and δy of x and y.

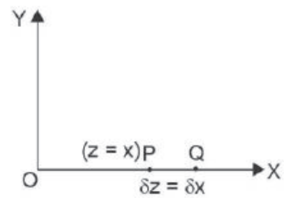
$$\therefore \quad f(z+\delta z) = (u+\delta u) + i(v+\delta v)$$

Now $\dfrac{f(z+\delta z) - f(z)}{\delta z} = \dfrac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z} = \dfrac{\delta u + i\delta v}{\delta z} = \dfrac{\delta u}{\delta z} + i\dfrac{\delta v}{\delta z}$

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \text{ or } f'(z) = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \qquad \ldots (1)$$

since δz can approach zero along any path.

(*a*) **Along real axis (x-axis)**



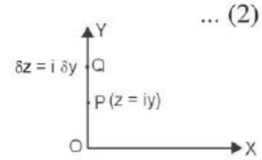
$z = x + iy$ but on x-axis, $y = 0$

$\therefore \quad z = x,$ $\delta z = \delta x, \ \delta y = 0$

Putting these values in (1), we have

$$f'(z) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \ldots (2)$$

(*b*) **Along imaginary axis (y-axis)**



$z = x + iy$ but on y-axis, $x = 0$

$z = 0 + iy$ $\delta x = 0, \ \delta z = i\delta y.$

Putting these values in (1), we get

$$f'(z) = \lim_{\delta y \to 0}\left(\frac{\delta u}{i\delta y} + \frac{i\delta v}{i\delta y}\right) = \lim_{\delta y \to 0}\left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \ldots (3)$$

If $f(z)$ is differentiable, then two values of $f'(z)$ must be the same.

Equating (2) and (3), we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

$\boxed{\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}$ **are known as Cauchy Riemann equations.**

9 SUFFICIENT CONDITION FOR $f(z)$ TO BE ANALYTIC

Theorem. *The sufficient condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are* (*i*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

(*ii*) $\dfrac{\partial u}{\partial x}, \ \dfrac{\partial u}{\partial y}, \ \dfrac{\partial v}{\partial x}, \ \dfrac{\partial v}{\partial y}$ ***are continuous functions of x and y in region R.***

Proof. Let $f(z)$ be a single-valued function having

$$\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial v}{\partial x}, \ \frac{\partial v}{\partial y}$$

at each point in the region R. Then the $C - R$ equations are satisfied.

By Taylor's Theorem:

$$f(z+\delta z) = u(x+\delta x,\ y+\delta y) + iv(x+\delta x,\ y+\delta y)$$

$$= u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + \ldots + i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + \ldots\right]$$

$$= [u(x,y) + iv(x,y)] + \left[\frac{\partial u}{\partial x}\cdot\delta x + i\frac{\partial v}{\partial x}\cdot\delta x\right] + \left[\frac{\partial u}{\partial y}\delta y + i\frac{\partial v}{\partial y}\cdot\delta y\right] + \ldots$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y + \ldots$$

(Ignoring the terms of second power and higher powers)

$$\Rightarrow \quad f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right).\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y \qquad \ldots (1)$$

We know $C - R$ equations *i.e.*,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Replacing $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y}$ by $-\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial u}{\partial x}$ respectively in (1), we get

$$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right).\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right).\delta y \qquad \text{(taking } i \text{ common)}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right).\delta x + \left(i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right).i\delta y = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right).(\delta x + i\delta y) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right).\delta z$$

$$\Rightarrow \quad \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

$$\Rightarrow \quad \boxed{f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

Proved.

Remember: **1.** If a function is analytic in a domain D, then u, v satisfy $C - R$ conditions at all points in D.

2. $C - R$ conditions are necessary but not sufficient for analytic function.

3. $C - R$ conditions are sufficient if the partial derivative are continuous.