Subgroup

(For B.A/B.Sc, Part –I, Hons. And Subsidiary Coursesof Mathematics)

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Subgroup

Defination 2.1: If a subset *H* of a group *G* is closed under the binary operation and if *H* with the induced operation from *G* is itself a group, then H is a subgroup of G. In other words H is subgroup of *G* if

(1) *H* is a subset of G.
(2) *H* itself satisfy all the axioms of group under the induced binary operation from *G,*

We denote this by $H \leq G$ or $G \geq H$. Also, $H < G$ or $G > H$ means that $H \leq G$ but $H \neq G$.

Remarks: (1) If G is any group, then the subgroup consisting of G itself is the improper subgroup of G. All other subgroups of G are proper subgroups. The subgroup {e} is the trivial subgroup of G. All other subgroups are nontrivial. The term, when referred to "G has no

(3) Thus the term “no nontrivial proper ***subgroups of G***" refers to the only ***subgroups*** of *G* being the ***trivial*** group {e} and the group G itself.

Examples: (1) The set of even integer form a subgroup of <Z, +> and <Z, +> form subgroup of <Q, +> which is subgroup of <R, +>.

(2) The set {1,-1,i,-i} is subgroup of group G={ ± 1, $\pm i$, $\pm j$, $\pm k$}.
(3) The nth roots of unity in C form a subgroup H of the group C of non zero complex numbers under multiplication.

Theorem2.1: A non empty subset *H* of a group *G* is a subgroup of *G* iff

(1) $a, b \in H \quad \Rightarrow \quad a * b \in H$
(2) $a \in H \quad \Rightarrow \quad a^{-1} \in H$

Proof: Suppose H is a subgroup of G, then H must be closed with respect to composition ‘*’ in G, i.e. $a, b \in H \quad \Rightarrow \quad a * b \in H$
Let $a \in H$ be any element. As H itself is a group, each element of H will possess inverse in it, i.e. $a \in H \quad \Rightarrow \quad a^{-1} \in H$.
Thus the condition is necessary.

Conversely ,Let the given condition (1) and (2) holds, we want to prove that H is subgroup of *G.* Since by condition (1), the closure property hold in *H.*

Again $a, b, c \in H \Rightarrow \quad a, b, c \in G$

$\Rightarrow \quad a * (\mathrm{b} * \mathrm{c}) = (\mathrm{a} * \mathrm{b}) * \mathrm{c}$ [Since *G* is group.]

Thus , $b, c \in H \Rightarrow a * (\mathrm{b} * \mathrm{c}) = (\mathrm{a} * \mathrm{b}) * \mathrm{c}.$ Hence associativity hold in *H.*

Further by condition (2) $a, a^{-1} \in H$ and so by condition (1)

$\Rightarrow \quad a * a^{-1} \in H \quad \Rightarrow \quad e \in H$, Thus *H* has identity element. Also by condition (2) inverse of every element of *H* exist in *H.* Thus *H* is a subgroup of *G.*

Theorem 2.2: A non empty subset *H* of a group *G* is a subgroup of *G* iff

$$\forall \quad a, b \in H \quad \Rightarrow \quad a * b^{-1} \in H$$

Proof: Let *H* be a subgroup of *G*. Then for, $a, b \in H$ we have $b^{-1} \in H$ and $a * b^{-1} \in H$ because *H* must be closed under the induced operation.

Conversely, suppose that *H* is nonempty subset of *G* and $a * b^{-1} \in H$ for all $a, b \in H$. Let $a \in H$. Then taking *b* = *a*, we see that $a * a^{-1} \in H \Rightarrow \quad a * a^{-1} = e \in H$. Taking *a* = *e,* and *b* = *a*, we see that $\Rightarrow \quad e * a^{-1} = a^{-1} \in H$. Thus *H* contains the identity element and the inverse of each element. For closure, note that for $a, b \in H$, we also have $a, b^{-1} \in H$ and thus

$$a * (b^{-1})^{-1} = a * b \in H.$$

Thus *H* satisfied all conditions of the group under induced operation of G. Hence *H* is subgroup of *G.*

Theorem 2.3: A non empty finite subset *H* of a group *G* is a subgroup of *G* iff

$$\forall \quad a, b \in H \quad \Rightarrow \quad a * b \in H$$

Proof: Assume that given condition hold i.e. $\forall \quad a, b \in H \quad \Rightarrow \quad a * b \in H$. By theorem 2.2, we need only show that $a^{-1} \in H$ whenever $a \in H$ to prove *H* is a subgroup of the group *G.* If $a = e$, then $a^{-1} = a$, and we are done. So suppose $a \neq e$. Consider the sequence $a, a^2, a^3,$ ...By closure, each of these elements are in *H.* Since *H* is finite, not all of these elements are distinct. Suppose $a^i = a^j$, $i > j$. Then $a^{i-j} = e$, and since $a \neq e$, i -j > 1. Thus, $a^{i-j-1} * a = e$, so $a^{i-j-1} = a^{-1}$. But $-j - 1 \geq 1$, so $a^{i-j-1} \in H$ and we are done.

Conversely, suppose H is a subgroup G, then by definition H itself a group under induced operation of G. Hence the closure property must satisfied i.e. $\forall \quad a, b \in H, \ a * b \in H$.

Theorem 2.4: If H and K are two subgroup of a group G, then $H \cap K$ is a subgroup of G.

Proof: Suppose H and K are two subgroup of a group G, then $e \in H \ , e \in K$. Thus $H \cap K \neq \emptyset$.

Now let $a, b \in H \cap K \quad \Rightarrow \quad a, b \in H$ and $a, b \in K$.

Since $a, b \in H$ and H is a subgroup of G, then $a * b^{-1} \in H$ (1)

Also $a, b \in K$ and K is a subgroup of G, then $a * b^{-1} \in K$ (2)

From (1) and (2) we have $a * b^{-1} \in H$ and $a * b^{-1} \in K$ i.e $a * b^{-1} \in H \cap K$.

Thus whenever $a, b \in H \cap K \quad \Rightarrow \quad a * b^{-1} \in H \cap K$. Therefore $H \cap K$ is a subgroup of G.