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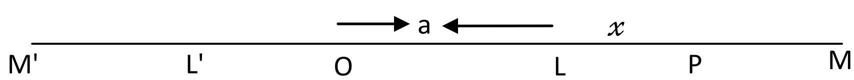
Program for B.Sc (Hons) Part-2

Topic :- Discussion of motion of Horizontal Elastic String.

Statement :- One end of an elastic string whose modulus of elasticity is λ, and whose natural length is 'a' is tied to fixed point on a smooth horizontal table and the other end is tied to a particle of mass 'm' lying on the table. The particle is pulled to distance where the extension of the string becomes 'b' and then let go. Now, we will discuss the motion and also prove that the period of one complete oscillation is

$$2\left(\pi+\frac{2a}{b}\right)\sqrt{\frac{am}{\lambda}}.$$

Verification:-



Suppose O be the fixed point on the smooth horizontal table MM' and suppose one end of an elastic string be fixed with O. Suppose OL= a be the natural length of the elastic string. Suppose a particle of mass m' be attached to the free end of the string i.e. at L and suppose the particle be pulled to M from L such that LM=b and then let go. Suppose P be the displaced position of the particle at time t such that LP=x. Here, x is the extension at time t.

Now, according to Hooke's Law, the tension T in the string P is

$\lambda\frac{x}{a}$ and this is the only force

acting on the particle in the direction of O. The equation of motion can be written

as $\quad m.\frac{d^2x}{dt^2}=-T,$

or $\quad m.\frac{d^2x}{dt^2}=-\left(\lambda.\frac{x}{a}\right)$

Here negative sign in the R.H.S is taken because x decreases as t increases.

$$\Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{\lambda}{ma}\right).x \quad \text{..............................} \quad (1)$$

Equation (1) is the standard equation of S.H.M about L and this equation hold good so long as the string is stretched. Therefore, the period of oscillation is

$$\frac{2\pi}{\sqrt{\frac{\lambda}{am}}} = 2\pi\sqrt{\frac{am}{\lambda}} \text{.....................................}(2)$$

At the point M, the velocity is zero and therefore LM will be the amplitude of the S.H.M

Now from equation – (1), we get

$$v.\frac{dv}{dx} = -\left(\frac{\lambda}{am}\right)x \text{ ; Since } \quad \frac{d^2x}{dt^2} = \frac{v.dv}{dx}.$$

$$\Rightarrow \quad v.dv = -\left(\frac{\lambda}{am}\right) \quad \text{on integration , we get}$$

$$\int v.dv = -\left(\frac{\lambda}{am}\right)\int x dx + C, \text{ where 'C' is the constant of integration.}$$

$$\Rightarrow \frac{v^2}{2} = -\left(\frac{\lambda}{am}\right).\frac{x^2}{2} + C \quad \text{..............................} \quad (3)$$

At the point M, t=o, x=b and v =o, so the equation (3) becomes

$$0 = -\frac{\lambda}{am}.\frac{b^2}{2} + C$$

$$\Rightarrow \quad C = \frac{\lambda b^2}{2am}$$

Putting this value in (3), we get

$$\frac{v^2}{2} = -\frac{\lambda}{am}.\frac{x^2}{2} + \frac{\lambda b^2}{2am}$$

$$\Rightarrow \quad v^2 = \frac{\lambda}{am}(b^2 - x^2)$$

$$\Rightarrow \quad v = \frac{dx}{dt} = -\sqrt{\frac{\lambda}{am}}.\sqrt{(b^2 - x^2)} \text{ ....}(4)$$

In the R.H.S, negative sign is taken as x decreases when t increases.

Equation – (4) gives the expression for the velocity of the particle at any time 't'

Where the string is stretched.

Characteristics of the motion :-

When the particle reaches at the point L, The velocity of the particle will be

$$v=-\sqrt{\frac{\lambda}{am}}..\sqrt{(b^2-0^2)}\ ;\text{ as}\qquad x=o$$

$$=-\sqrt{\frac{\lambda}{am}}.b$$

In this case the string gets its natural length, therefore tension T=O at that point and the particle moves towards O with a uniform velocity $\sqrt{\frac{\lambda}{am}}$ b.

After this, the string becomes slack, so that the equation of motion (1) does not hold any more. Actually, the particle moves like a free particle on the smooth table without any force along LO L', with a constant velocity

$\sqrt{\frac{\lambda}{am}}$. b acquired at L, till it reaches L' ,

Where OL' =OL=a , The time taken to reach L' from L is given by

$$\frac{2a}{b\sqrt{\frac{\lambda}{am}}}=\frac{2a}{b}\sqrt{\frac{am}{\lambda}}.$$

After L', the string again extends and the tension comes into play and the motion is again simple harmonic the particle comes to rest at a point M' where L'M'=LM and then retraces its path in the same way.

The motion is simple harmonic for the amplitude ML or L'M' and the period is

$$2\pi\sqrt{\frac{am}{\lambda}}.$$

The time taken from L to L' and L' to L

$$=\frac{4a}{b}\sqrt{\frac{am}{\lambda}}$$

Therefore the time of one complete oscillation

$$=2\pi\sqrt{\frac{am}{\lambda}}+\frac{4a}{b}\sqrt{\frac{am}{\lambda}}$$

$$=2\left(\pi+\frac{2a}{b}\right)\sqrt{\frac{am}{\lambda}}$$

Hence the result.