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Program for B.Sc (Hons) Part-3

Topic: - Automorphism.

Definition of automorphism:- Let G be a group. Now we define a mapping

f: G→G such that

i. f is one-one

ii. f is onto

iii. f preserves operations, then f is called automorphism on G.

Moreover an isomorphism of a group G onto itself is called an automorphism of the group.

Ex. 1. Let G be a group. Then the identity mapping I:G→G is an automorphism of G.

Ex. 2. Let C be additive group of complex numbers. Now we define a mapping f: C→C given by f (x+ iy)= x-y , where (x-iy) ϵ C. Here f is an automorphilsm on C.

Verification: - Given mopping is

f:C →C , where C is the additive group of complex numbers

Let $z = x+iy$. So $\bar{z} = x - iy$.

Since $f(x+iy) = x- iy$

$\Rightarrow f(z) = \bar{z}$ for all $\bar{z} \in C$.

Now, we have to prove that f is an automorphism for this, we prove that followings:-

i. f is one-one : → Let $z_1, z_2 \in C$.

Then $f(z_1) = f(z_2) \quad \Rightarrow \bar{z}_1 = \bar{z}_2$

$\Rightarrow \overline{(\bar{z}_1)} = \overline{(\bar{z}_2)}$

$\Rightarrow z_1 = z_2$

$\Rightarrow$ f is one − one.

ii. f is onto – Let z=(x+iy) be an element of C, then (x-iy) is also an element of c.
Also f [(x-iy)] = x+iy.
So, f is onto.

iii. f preserves operations :- Let $z_1, z_2 \epsilon$ C then $(z_1 + z_2) = \overline{(z_1 + z_2)}$ $= \bar{z}_1 + \bar{z}_2 = f(z_1) + f(z_2)$ this shows that f is operation preserving.

Hence f is an atomorphism on C.

Theorems on automorphism

1. The set of all automorphism of a group forms a group with respect to composite of function as the compositions.

Proof: - Suppose A (G) be the set of all automorphism of a group G. Then A (G) = {f: f is an automorphism of G}.

Now, we have to show that A (G) is a group with respect to composite of function as composition. For this, we prove the followings:-

i. **Closure property**: suppose f, g ∈ A (G). Then f, g are one one mapping of G onto itself. So gf is also a one-one mapping of G onto itself.

Suppose a, b be any two elements ∈G. Then we have

gf (ab) = g {f (ab)} = g[f(a)f(b)]; since f is automorphism

= g [f (a)] g[f(b)]; g is an isomorphism

= [gf (a)] [gf(b)].

So, gf is also an automorphism of G.

Thus A (G) is closed w.r.t composite mapping.

2. **Associative Property**: As we know that composite of any orbitray mappings f, g, h are associative therefore composite of automorphism is also associative.
3. **Existence of identity**: The identity function I on G is clearly one-one and if a, b ϵ G, there I (ab)=ab=I (a) I (b).
Therefore I on G is also an automorphism of G.
This Iϵ A (G) and if f ϵ A (G), the if =f=f I.

4. **Existence of inverse:** Let $f \in A(G)$.

Since f is a 1-1 mapping of G onto itself, therefore f^{-1} is also 1-1 mapping of G onto it self

Now, we prove that f^{-1} is also an automorphism of G.

Let, a, b $\in G_2$ (Second group). Then there exists $a', b' \in G_1$ (frist group) such that

$$f^{-1}(a)=a' \text{ and } f^{-1}(b)=b'$$

Now, $f^{-1}(a) = a' \Rightarrow f(a') = a$ and $f^{-1}(b) = b' \Rightarrow f(b') = b$

We have $f^{-1}(ab) = f^{-1}[f(a') f(b')]$

$= f^{-1}[f(a'b')]$; f is an isomorphism

$= a' b' = f^{-1}(a)f^{-1}(b)$.

Therefore f^{-1} is an automorphism of G and thus $f \in A(G)$. So, each element A (G) possesses inverse.

So, A (G) is a group w.r.t composite f^n.