Total Differential and Exact Differential

(For B.Sc./B.A. Part-I, Hons. Course of Mathematics)

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Contents

1. Total Differential

Theorem : If u be a function of two independent variables x, y and if $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are continuous in the neighborhood of the point (x, y), then

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

Proof : Let $u = f(x, y)$ be a function of two independent variables $x, y.$

Let $\delta x, \delta y$ denote small increments in x, y respectively and let δu denote the corresponding increment in $u.$ Then, we have

$$u + \delta u = f(x + \delta x, y + \delta y).$$

Therefore

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)$$

or
$$\delta u = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \delta x + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \delta y \qquad .....(1)$$

By Mean Value Theorem, we have

$$\left.\begin{aligned} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} &= \frac{\partial}{\partial x} f\left(x + \theta_1 \delta x, y + \delta y\right) \\ \text{and} \quad \frac{f(x, y + \delta y) - f(x, y)}{\delta y} &= \frac{\partial}{\partial y} f\left(x, y + \theta_2 \delta y\right) \end{aligned}\right\} \qquad ....(2)$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1.$

Substituting the values from equation (2) in equation (1), we get

$$\delta u = \left[\frac{\partial}{\partial x} f\left(x + \theta_1 \delta x, y + \delta y\right)\right] \delta x + \left[\frac{\partial}{\partial y} f\left(x, y + + \theta_2 \delta y\right)\right] \delta y \qquad ......(3)$$

Since $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous in the neighborhood of the point (x, y), we have

$$\left.\begin{aligned} \frac{\partial}{\partial x} f\left(x + \theta_1 \delta x, y + \delta y\right) &= \frac{\partial}{\partial x} f\left(x, y\right) + \varepsilon_1 \\ \text{and} \quad \frac{\partial}{\partial y} f\left(x, y + + \theta_2 \delta y\right) &= \frac{\partial}{\partial y} f\left(x, y\right) + \varepsilon_2 \end{aligned}\right\} \qquad ....(4)$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\delta x, \delta y \to 0$.

Substituting the values from equation (4) in equation (3), we get

$$\delta u = \left[\frac{\partial}{\partial x} f(x, y)\right]\delta x + \left[\frac{\partial}{\partial y} f(x, y)\right]\delta y + \varepsilon_1 \delta x + \varepsilon_2 \delta y \qquad ......(5)$$

In the limiting case, when $\delta x \to 0,\ \delta y \to 0,\ \delta u \to 0$, it follows from equation (5) that

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

i.e.,

$$\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy}$$

Note : The above result can be extended to a function of any number of variables, i.e.,

(i) if u be a function of three independent variables x, y, z and if $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ are continuous in the neighborhood of the point (x, y, z), then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz.$$

(ii) if u is a function of n independent variables $x_1, x_2, x_3, \ldots\ldots, x_n$ and if each of the partial derivatives $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \ldots\ldots, \frac{\partial u}{\partial x_n}$ is continuous in the neighborhood of the point $(x_1, x_2, x_3, \ldots\ldots, x_n)$, then

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \frac{\partial u}{\partial x_3}dx_3 + \ldots\ldots + \frac{\partial u}{\partial x_n}dx_n.$$

2. Exact Differential

Definition : An expression of the type $M\,dx + N\,dy$, where M and N are functions of x and y or constants, is called an exact differential if it can be reduced to the form du, where u is a function of x and y.

Theorem : The necessary and sufficient condition for the expression $M\,dx + N\,dy$ to be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Proof : Case 1. The condition is necessary.

Let the given expression $M\,dx + N\,dy$ be an exact differential.

To prove : $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Since the given expression $M\,dx + N\,dy$ is an exact differential, therefore, by definition, there exists u, a function of x and y such that

$$M\,dx + N\,dy = du.$$

This $\Rightarrow M\,dx + N\,dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad .....(1)$

$$\left(\because du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)$$

Equating coefficients of dx and dy from both sides of equation (1), we get

$$M = \frac{\partial u}{\partial x} \text{ and } N = \frac{\partial u}{\partial y}$$

This $\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial y \partial x}$ .....(2)

and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial x \partial y}$ ......(3)

We know that

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \quad .....(4)$$

Equations (2), (3), (4) $\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Case 2. The condition is sufficient.

Now we suppose that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad ......(5)$$

where M and N are functions of x, y.

To prove : $M\,dx + N\,dy$ is an exact differential.

By definition, $M\,dx + N\,dy$ will be an exact differential if we can find a function u of x, y such that

$$M\,dx + N\,dy = du \quad ......(6)$$

Let us suppose that there exists a function u of x, y satisfying equation (6). Then we have

$$M\,dx + N\,dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\left(\because du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)$$

Equating coefficients of dx and dy from both sides of the above equation, we get

$$M = \frac{\partial u}{\partial x} \quad ....(7)$$

and $N = \dfrac{\partial u}{\partial y}$ ....(8)

Our purpose is to find the function u of x, y.

Equation (7) $\Rightarrow u(x, y) = \displaystyle\int_{y\,\text{constant}} M\,dx + \phi(y)$ ......(9)

where ϕ is a function of y.

Now, it remains to find the function ϕ only.

Equation (9) $\Rightarrow \dfrac{\partial u}{\partial y} = \dfrac{\partial}{\partial y}\left(\displaystyle\int_{y\,\text{constant}} M\,dx\right) + \dfrac{d\phi}{dy}$

$$\Rightarrow N = \frac{\partial}{\partial y}\left(\int_{y\,\text{constant}} M\,dx\right) + \frac{d\phi}{dy} \qquad \left(\because \frac{\partial u}{\partial y} = N,\ \text{from equation (8)}\right)$$

$$\Rightarrow \frac{d\phi}{dy} = N - \frac{\partial}{\partial y}\left(\int_{y\,\text{constant}} M\,dx\right)$$

$$\Rightarrow \phi(y) = \int\left[N - \frac{\partial}{\partial y}\left(\int_{y\,\text{constant}} M\,dx\right)\right]dy \quad .....(10)$$

Since ϕ is a function of y only, therefore equation (10) is true only if $N - \dfrac{\partial}{\partial y}\left(\displaystyle\int_{y\,\text{constant}} M\,dx\right)$ is independent of x. To check this, we have

$$\frac{\partial}{\partial x}\left[N - \frac{\partial}{\partial y}\left(\int_{y\,\text{constant}} M\,dx\right)\right] = \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x\,\partial y}\left(\int_{y\,\text{constant}} M\,dx\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial y\,\partial x}\left(\int_{y\,\text{constant}} M\,dx\right) \qquad \left(\because \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\int_{y\,\text{constant}} M\,dx\right)\right]$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= 0 \qquad \left(\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},\ \text{from equation (5)}\right)$$

This $\Rightarrow N - \dfrac{\partial}{\partial y}\left(\displaystyle\int_{y\,\text{constant}} M\,dx\right)$ is independent of x.

$\Rightarrow$ The function ϕ is a function of y only.

(using equation (10))

Substituting the value of $\phi(y)$ from equation (10) in equation (9), we get

$$u(x, y) = \int_{y\,\text{constant}} M\,dx + \int\left[N - \frac{\partial}{\partial y}\left(\int_{y\,\text{constant}} M\,dx\right)\right]dy \quad .....(11)$$

If the function u of x, y is defined by equation (11), then equation (6) holds good.

This $\Rightarrow M\,dx + N\,dy$ is an exact differential.

(using equation (6))

Example 1 : Prove that the expression $(2x+y-3)\,dx+(x-4y+1)dy$ is an exact differential.

Solution : We know that the necessary and sufficient condition for the expression

$$M\,dx+N\,dy \qquad .....(1)$$

to be an exact differential is

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \qquad ......(2)$$

The given expression is

$$(2x+y-3)\,dx+(x-4y+1)dy \qquad .....(3)$$

Comparing the expression (3) with the expression (1), we get

$$M=2x+y-3 \text{ and } N=x-4y+1$$

This $\Rightarrow \frac{\partial M}{\partial y}=1$ and $\frac{\partial N}{\partial x}=1.$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$\Rightarrow$ The given expression satisfies condition (2).

$\Rightarrow$ The given expression is an exact differential.

Example 2 : Prove that the expression $(x^2+y^2)\,dx-2xy\,dy$ is not an exact differential.

Solution : We know that the necessary and sufficient condition for the expression

$$M\,dx+N\,dy \qquad .....(1)$$

to be an exact differential is

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \qquad ......(2)$$

The given expression is

$$(x^2+y^2)\,dx-2xy\,dy \qquad .....(3)$$

Comparing the expression (3) with the expression (1), we get

$$M=x^2+y^2 \text{ and } N=-2xy$$

This $\Rightarrow \frac{\partial M}{\partial y}=2y$ and $\frac{\partial N}{\partial x}=-2y.$

$$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$\Rightarrow$ The given expression does not satisfy condition (2).

$\Rightarrow$ The given expression is not an exact differential.

Exercises

1. Prove that the expression $\sin y\cos x\,dx+\sin x\cos y\,dy$ is an exact differential.
2. Prove that the expression $y\,dx-x\,dy$ is not an exact differential, although $\frac{1}{y^2}(y\,dx-x\,dy)$ is an exact differential.