Elementary properties of group

(For B.A/B.Sc, Part –I, Hons. And Subsidiary Coursesof Mathematics)

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Property 1: Prove that a group has unique identity element.

Proof: Suppose <G,*> be group having two identity element *e* and *f*. Since e and f are identity element, therefore

$$a * e = e * a = a$$

$$a * f = f * a = a \qquad \forall \ a \epsilon \ G$$

Now $e * f = f$ by using right identity and *e* as identity element. (1)

And $e * f = e$ by using left identity and *f* as identity element. (2)

From (1) and (2) we have

$$e = f$$

Thus identity element of group is unique.

Property 2: Prove that in a group inverse of element a is unique.

Proof: Suppose <G,*> be group and a be any element of group G. Let *b* and *c* be two inverse element *a*. Since *b* and *c* are inverse of *a*, therefore

$$a * b = b * a = e \qquad (1)$$

$$a * c = c * a = e \qquad (2)$$

where e is identity element of G.

Now $b = b * e$ (As *e* is identity element of G)

$= b * (a * c)$ (By equation (2))

$= (b * a) * c$ (By associative law)

$= e * c$ (By using equation (1))

$= c$

$\Rightarrow \quad b = c$

Thus inverse of a element is always unique.

Property 3: Prove that in a group G, $(a * b)^{-1} = b^{-1} * a^{-1}$.

Proof: Let <G,*> be group. Let a and b be any two element of the group G.

As $a, b \in G \implies c \in G,$ where $c = a * b$ (By closure property of group)

Also $a, b \in G \implies b^{-1}, a^{-1} \in G$ (As inverse of every element exist in G)

$\implies d = b^{-1} * a^{-1} \in G$ (By closure property of group)

Now consider $c * d = (a * b) * d$

$= a * (b * d)$

$= a * (b * (b^{-1} * a^{-1}))$

$= a * ((b * (b^{-1}) * a^{-1}))$

$= a * (e * (a^{-1}))$

$= a * a^{-1} = e$ (1)

Also $d * c = d * (a * b)$

$= (d * a) * b$

$= ((b^{-1} * a^{-1}) * a) * b)$

$= (b^{-1} * (a * a^{-1}) * b)$

$= b^{-1} * (e * b)$

$= b^{-1} * b = e$ (2)

Thus by equation (1) and (2) , we have

$$d * c = e = c * d$$

$$\implies c^{-1} = d$$

$$\Longrightarrow \qquad (a*b)^{-1} = b^{-1}*a^{-1}$$

Hence the result.

Theorem 1.1: Let *G* be group and *a, b, c* elements of *G*. Then

$a*b = a*c \quad \Rightarrow \quad b = c$ (left cancellation law)

$b*a = c*a \quad \Rightarrow \quad b = c$ (Right cancellation law)

Proof: Since *a* is an element of *G* and *G* is a group, therefore $a^{-1} \in G$.

Now consider $a*b = a*c$

Pre multiply both side by a^{-1} , we have

$$a^{-1}*(a*b) = a^{-1}*(a*c)$$

$$\Longrightarrow \quad (a^{-1}*a)*b = (a^{-1}*a)*c$$

$$\Longrightarrow \quad e*b = e*c$$

$$\Longrightarrow \quad b = c$$

Thus $a*b = a*c \quad \Longrightarrow \quad b = c$

Similarly consider $b*a = c*a$

Post multiply both side by a^{-1} , we have

$$(b*a)*\ a^{-1} = (c*a)*\ a^{-1}$$

$$\Longrightarrow \quad b*(a*a^{-1}) = c*(a*a^{-1})$$

$$\Longrightarrow \quad b*e = c*e$$

$$\Longrightarrow \quad b = c$$

Thus $b*a = c*a \quad \Longrightarrow \quad b = c$

Theorem 1.2. Let G be a group and a,b elements of G. Then the equation $a * x = b$ and $y * a = b$, have unique solution in G.

Proof: Since G is group and $a, b \in G$.

$\Longrightarrow \quad a^{-1}, b \in G$ [By inverse property of group G]

$\Longrightarrow \quad a^{-1} * b \in G$ [By closure property of group G]

Now, we show that element $x = a^{-1} * b \in G$, is solution of the equation $a * x = b$.

Consider $a * x = a * (a^{-1} * b)$

$= (a * a^{-1}) * b$

$= e * b$

$= b$.

Thus the equation $a * x = b$ has solution in G.

To prove uniqueness, Let us assume that equation $a * x = b$ have two solution say x_1 and x_2.

$\Longrightarrow \quad a * x_1 = b$ (1)

$\Longrightarrow \quad a * x_2 = b$ (2)

From (1) and (2), we have

$a * x_1 = a * x_2$

By left cancellation law we have,

$x_1 = x_2$

Thus equation $a * x = b$ has unique solution in G.

Similarly we can prove that equation $y * a = b$ has unique solution in G.

Defination1.3: A non empty set G is said to be semi-group if in G there defined a operation '*' such that:

(1) For all *a*, *b* in *G*, the result of the operation, *a * b*, is also in *G*. **[Closure law]**

(2) For all *a*, *b* and *c* in *G*, then *(a * b) * c = a * (b * c)*. **[Associative Law]**

Theorem 1.3: Let $< G,*>$ be a semi group. Then G is group under binary operation '*' if

(1) $\exists$ an element $e \in G$ such that $e * a = a$ for all $a \in G$.

(2) For each $a \in G$, $\exists$ an element $a^{-1} \in G$ such that $a^{-1} * a = e$.

Proof : We first show that if left identity and left inverse hold in *G,* then left cancellation law must be hold in *G* *i.e.* if $a, b, c \in G$ then $a * b = a * c \Rightarrow b = c$.

Since $a \in G \Rightarrow \exists$ an element $a^{-1} \in G$ such that $a^{-1} * a = e$.

Now consider $a * b = a * c \Rightarrow a^{-1} * (a * b) = a^{-1} * (a * c)$

$$\Rightarrow (a^{-1} * a) * b = (a^{-1} * a) * c$$

$$\Rightarrow e * b = e * c$$

$$\Rightarrow b = c.$$

Thus left cancellation law hold in *G*. Now in order to prove the theorem we have to show that

(1) Left identity is also right identity.

(2) Left inverse of a element is also its right inverse.

Let $a \in G$ be any element and e be the left identity and a^{-1} be left inverse of a.

Therefore $e * a = a$ and $a^{-1} * a = e$.

Now consider $a^{-1} * (a * e) = (a^{-1} * a) * e = e * e = e = a^{-1} * a$.

Thus $a^{-1} * (a * e) = a^{-1} * a$, then by left cancellation law we have,

$a * e = a$, which is right identity.

Also $a^{-1} * (a * a^{-1}) = (a^{-1} * a) * a^{-1} = e * a^{-1} = a^{-1} = a^{-1} * e$

Thus $a^{-1} * (a * a^{-1}) = a^{-1} * e$, then by left cancellation law we have,

$a * a^{-1} = e$, which right inverse of a.

Thus <$G, *$> is group.

Note: (1). Similarly the above theorem 1.3 can be proved for right identity and right inverse.

(2). The above result may not true for left identity and right inverse or right identity and left inverse.

Ex.7 Let G be finite set having at least two elements. Define binary operation as '*' on G as

$a * b = b$ for all $a, b \in G$, then clearly associative law hold in G. Let $e \in G$ be any fixed element , then $e * a = a$ for all $a \in G$. Thus e is a left identity . Again $a * e = e$ for all $a \in G$. Thus e is right inverse of element a exist in G.

Now we show that G is not a group. If possible suppose G is a group. Let $a, b, c \in G$ be any elements of G such that $a \neq c$.

Now by given operation we have

$a * b = b$

$c * b = b$.

Thus $a * b = c * b$ by cancellation law we have $a = c$. But here we have $a \neq c$. Thus cancellation law does not hold in G. Hence by theorem(1.1) G is not a group under given operation.

Theorem 1.4: A non empty set G together with binary operation '*' is group if and only if

(3) $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$.

(4) For any $a, b \in G$, the equations $a * x = b$ and $y * a = b$, have unique solution in G.

Proof: Since <G *> is a Semi -group therefore it is closed under the operation '*' and associative property for the operation also holds in it. So in order that G become a group, we need two more properties to hold in G:

(1) Existence of the right identity element.

(2) Right inverse of every element of G exist in G.

To prove right identity: As we know that the two given equations

$a * x = b$ (1)

$y * a = b$ (2)

have unique solutions in *G* for any two elements *a, b* in *G* . In particular equation (1) has solutions for $a = b$ i.e. the equations $a * x = a$. Let $x = e \in G$ be the solution of this equation. Thus we have, $a * e = a$. Now e be the right identity if it hold for any element $b \in G$.

Consider $b * e = (y * a) * e$

$$= y * (a * e)$$

$$= y * a$$

$$= b$$

$\Rightarrow$ $b * e = b$, for any element $b \in G$. Thus *e* is the right identity of *G*.

To prove existence of right inverse: Since equation (1) has solution for all $a, b \in G$. As $e \in G$,therefore equation $a * x = e$ must have solution in G. Thus $x' \in G$ such that $a * x' = e$. Which implies $x' \in G$ is right inverse of *a*. Thus every element of *G* have inverse in *G*.

Theorem 1.5. A finite semi group in which cancellation laws hold is a group.

Proof: Let $G = \{a_1, a_2, \dots, a_n\}$be finite semi group in which cancellation law hold and all the elements are distinct. Let $a \in G$ be any element, then by closure property, the elements $aa_1, aa_2, \dots, aa_n$ are all in *G*.

If possible suppose any two of these elements are equal say $aa_i = aa_j$ for some $i \neq j$, then by cancellation law we have $a_i = a_j$. But $a_i \neq a_j$ as $i \neq j$. Hence no two elements of $aa_1, aa_2, \dots, aa_n$ can be equal. These elements are distinct member of *G*. Thus if $b \in G$ be any element then $b = aa_i$ for some *i,* that is for any $a, b \in G$ the equation *ax=b* has a solution in *G*. Similarly we can show that the equation *ya= b* has a solution in *G*. Hence by theorem 1.4 G is group.

Ex.8 Show that a group in which every element is its own inverse is abelian.

Solution: Let G be group in which every element is its own inverse. Let $a, b \in G$ be any two element s, then we have $a = a^{-1}$ and $b = b^{-1}$.

Also $a, b \in G$, then by closure law we have $ab \in G$.

There fore $ab = (ab)^{-1}$ (1)

But $(ab)^{-1} = b^{-1}a^{-1}$ (2)

From (1) and (2) we have

$ab = b^{-1}a^{-1} = ba$ for all $a, b \in G$. Thus G is abelion group.

Ex. 9. Show that $(ab)^2 = a^2b^2$ for all choices of $a, b \in G$ in agroup iff G is abelian.

Solution: Suppose G is abelian, then, $ab = ba$

Now $(ab)^2 = (ab)(ab)$

$= a(ba)b$

$= a(ab)b$

$= a^2b^2$

Conversely suppose $(ab)^2 = a^2b^2$, we need to show that $ab = ba$

$(ab)^2 = (ab)(ab)$

Also $a^2b^2 = (aa)(bb)$

Thus $(ab)(ab) = (aa)(bb)$, Now by cancellation law we have

$ab = ba$

Therefore G is abelian group.