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Programme for B.Sc. (Hons) Part-1

Notion of Countability of a Set

(Cardinal number of a set) :- The cardinal no of a finite set is the total no of distinct elements present in the set. It is denoted by n (A) or |A| and read as 'the number of elements of the set'.

Examples :-

(i) $A = \{1,3,5\}$. $\quad n(A) = 3$.

(ii) $B = \{a, e, i, o, u\}$ $\quad n(B) = 5$

(iii) Set $C = \{ \quad \}$.

The set C h is no elements so it is null set. n (C) = O.

(iv) $IN = \{1,2,3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.\}$

The set of natural number is not a finite set. So, we can't get the cardinal no. of C.

Thus (N)= undefined.

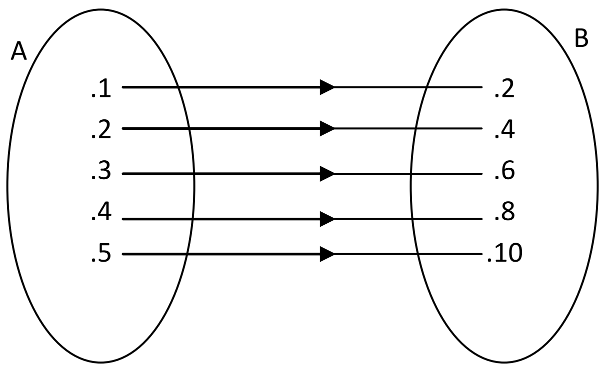
Note:- (i) Cardinal no is only defined for a finite set.

(ii) Since the null set has no element. So its $(\emptyset)=0$.

(iii) The cardinal no. of a infinite set is not defined.

Equipotent set :- Let A and B are the two sets such that there exists a one -one onto mapping $\mathcal{F}$ from A to B. That is $\mathcal{F}$: A→B is bijective mapping . Now A and B are called equipotent sets to each other equipotent sets are also called equivalents sets.

Example of equipotent sets:-



(i) Two sets A and B are equipotent if and only if they have the same cardinal.

(ii) Two infinite sets may have distinct cardinal number if they are not equipotent. Therefore the set of whole numbers, which is an infinite set, does not have the same cardinal number as the set of real numbers.

(iii) All of the infinite sets that are equipotent to the set of whole number is called countably infinite sets. This is the case for ℤ and Q.
On other hand, The set of real numbers is a countably infinity set.

Finite sets :- Finite sets are either empty or have n elements. If a set has n elements, there exists a one to one correspondence with the set of natural numbers, {1,2,3………, n} where n e IN.

Ex :- {a,b,c} can be put into a one to one correspondence with {1,2,3}. One such function is p→1, q →2 , r→3. Finite sets are also called countably finite sets.

Countably infinite set – A set A is countably infinite if and only if set A has the same cardinality as IN i.e. The set of natural numbers. If the set A is coutaby infinite, then |A|=|IN|. Furthermore, we designate the cordinality of contably infinite sets as x_0 ('aleph not). |A|=|IN| = X_0.

Example of countably infinite set :- The set of integers ℤ is countably infinite.

Countable :- A set is countable if and only if it is finite or countably infinite.

Uncountably infinite :- A set which is not countable is known as uncountable or uncountably infinite. Ex – IR, The set or real numbers is uncountably infinite.

Countable set :- A set equipotent to the set of natural numbers and hence of the some cardinality, is called countable.

The union and Cartesian product of two countable sets is countable; the union of a conuntable family of countable sets is also Countable. Moreover, a "countably infinite or "denumerable" set and "countable" set means finite or **countably infinite:**- that is a set of the same cardinality as some subset of the natural numbers.

An uncountable set is one which is not countable. For ex:- the set of real number is uncountable infinite.

Countable Set :- A set equipotent to the set of natural numbers and hence of the same cardinality is called countable.

Ex :- The set of integers, the set of rational numbers or the set of algebraic numbers are countable.

The union and Cartesian product of two countable sets is countable, the union of a countable family of countable sets is also countable.

Moreover, a "countably infinite" or "denumenrable" set and a "Countable" set means finite or Count ably infinite : that is, a set of the same cardinality as some subset of the natural numbers.

An uncountable set is one which is not countable. For ex :- the set of real numbers is uncountable (by cantor's theorem).

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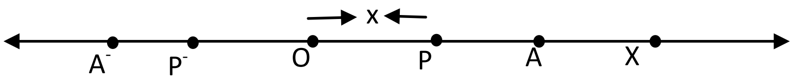
Programme for B.Sc. (Hons) Part-2

Simple Harmonic Motion

Simple harmonic motion is that motion in which a particle moves on a straight line in such a way that its acceleration at any point of its motion is always directed towards some fixed point on the line and is directly proportional to its displacement from that fixed point on the line and is directly proportional to its displacement from that fixed point.



Let a particle moves on the above state line A'A. Here 'O' is the fixed point and OA=a. Now, let P be the position of the particle at any time't' and ' x ' be the distance of P from O. Again, let the acceleration of the particle at the point P is $\frac{d^2x}{dt^2}$ or $V.\frac{dv}{dx}$ in the direction OP.

According to the definition of S.H.M as mentioned above, the equation of motion of the particle can be written as

$$\frac{d^2x}{dt^2} = -\mu x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Or, $v.\frac{dv}{dx} = -\mu x$

Or, $v.dv = -\mu x dx$

On integration, we get

$$\int v.dv = -\mu \int x dx + C, \qquad c = \text{Constant of integration}$$

$\Rightarrow \frac{v^2}{2} = -\mu\frac{x^2}{2} + C$

$\Rightarrow v^2 = -\mu x^2 + 2C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

In the beginning of the motion, $\frac{dx}{dt} = 0$ when x =a.

Therefore, equation (2) becomes

$0 = -\mu a^2 + 2c \Rightarrow 2c = \mu a^2$

By putting this value in eq.(2) , we get

$$v^2 = -\mu x^2 + \mu a^2 = \mu(-x^2 + a^2) = \mu\,(a^2 - x^2)$$

$$\Rightarrow v = \pm\sqrt{\mu(a^2 - x^2)} = \pm\sqrt{\mu}.\sqrt{(a^2 - x^2)}$$

$$\Rightarrow \frac{dx}{dt} = \pm\sqrt{\mu}.\sqrt{(a^2 - x^2)}$$

Here, Velocity is negative since acceleration and displacement have opposite direction.

$$\Rightarrow \frac{dx}{dt} = -\sqrt{\mu}.\sqrt{(a^2 - x^2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

$$\Rightarrow \frac{-dx}{\sqrt{(a^2 - x^2)}} = \pm\sqrt{\mu}.dt$$

$$\Rightarrow -\int \frac{dx}{\sqrt{(a^2 - x^2)}} + K = \sqrt{\mu}.dt, \text{where}$$

K is the constant of integration

Put $\qquad x = a\cos\theta$;

$\therefore \qquad d\,x = -a\sin\theta\, d\theta$

Therefore $\int \frac{a\sin\theta}{\sqrt{a^2 - a^2\cos^2\theta}} d\theta + K = \sqrt{\mu}\ t$

$$\Rightarrow \int \frac{\cancel{a\sin\theta}}{\cancel{a\sin\theta}}.d\theta + k = \sqrt{\mu}\ t$$

$$\Rightarrow \theta + k = \sqrt{\mu}\ t$$

$$\Rightarrow \cos^{-1}\left(\frac{x}{a}\right) + K = \sqrt{\mu}\ t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

When $t=o$, $x=a$; $k=o$. So, eq. (4) becomes, $\cos^{-1}\left(\frac{x}{a}\right)\sqrt{\mu}\ t$

$$\Rightarrow \frac{x}{a} = \cos\sqrt{\mu}\ t \Rightarrow x = a\ \cos\sqrt{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

Equation (5) is the expression for displacement at any time't'

Amplitude :- the quantity OA=a is the largest displacement of the particle form 'O' is called the centre of oscillation.

Therefore by amplitude we mean the maximum distance through which the particle vibrates under simple harmonic motion on either side of the centre of oscillation.

Periodic time or period of oscillation :- The time for one complete oscillation, that is, from A to A' and back again, is called the periodic, time or period of oscillation, which is equal to 4t. Now the time from A to O is obtained by putting x=o is (5), gives

$$0 = a\cos(\sqrt{\mu}\,t).\text{but } a \neq o, \text{so that } \cos(\sqrt{\mu}\,t) = o.$$

$$\text{that is } \sqrt{\mu}\,t = \frac{\pi}{2}$$

$$\text{Or } t = \frac{\pi}{2\sqrt{\mu}}$$

So, Periodic time, T = 4t

$$= 2\pi \times \frac{\pi}{2\sqrt{\mu}} = \frac{2\pi}{\sqrt{\mu}}$$

$$\boxed{T = \frac{2\pi}{\sqrt{\mu}}}$$

T is independent of amplitude.

Frequency :- The number complete oscillation per unit time is called the frequency of oscillation or simply frequency. Thus, if n is the frequency, then

$$n = \frac{1}{T} = \frac{1}{2\pi/\sqrt{\mu}} = \frac{\sqrt{\mu}}{2\pi}$$

$$\Rightarrow \quad n = \frac{\sqrt{\mu}}{2\pi}.$$

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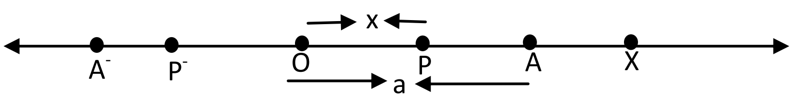
Topic – Motion under Inverse Squre Law

A particle movie in a straight line with an acceleration with is always directed towards a fixed point O and Vavies inversely as the square of its distance from O.



Suppose 'O' be the fixed point and A, the point from which the particle starts from rest such that OA=a.

Again suppose P be the position of the particle after time't' from the instant of the start of the motion and let OP $=x$. According to the Inverse square law the equation of motion of the particle can be written as.

$$\frac{d^2x}{dt^2} = -\mu.\frac{1}{x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\Rightarrow 2\frac{dx}{dt}.\frac{d^2x}{dt^2} = -\mu.\frac{1}{x^2}.2\frac{dx}{dt}$$

$$\Rightarrow \int 2\frac{dx}{dt}.\frac{d^2x}{dt^2} = \int -\mu.\frac{1}{x^2}.2\frac{dx}{dt}$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} + C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where C is the arbitrary constant of integration

$\because$ t =o , $\frac{dx}{dt} = 0$ and $x = a$, eq. (2) become

$$C = \frac{-2\mu}{a}.$$

Putting this value in (2) , we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} - \frac{2\mu}{a}$$

$$\Rightarrow \quad \left(\frac{dx}{dt}\right)^2 = 2\mu\left(\frac{a-x}{ax}\right)$$

$$\Rightarrow \quad \left(\frac{dx}{dt}\right)^2 = \sqrt{\left(\frac{2\mu}{a}\right)}\sqrt{\frac{a-x}{x}} \quad \text{...} (3)$$

Put $x = a\ \cos^2\theta\,.\, so\ dx = -2a\ cos\theta sin\theta d\theta.$

Now, the equation (4) becomes

$$\sqrt{\left(\frac{2\mu}{a}\right)}.t = \int \frac{\cos\theta}{\sin\theta}\ .2a\ cos\theta sin\theta d\theta.$$

$$\Rightarrow \quad \sqrt{\left(\frac{2\mu}{a}\right)}.t = a\ \int 2\cos^2\theta . do + D$$

$$\Rightarrow \quad \sqrt{\left(\frac{2\mu}{a}\right)}.t = a\ \int (1 + \ \cos^2\theta)\ do + D$$

$$\because x = a\cos^2\theta,\ \ so\ \ \theta = \cos^{-1}\sqrt{\frac{x}{a}},\ \ \cos\theta = \sqrt{\frac{x}{a}},$$

$$\sin\theta = \sqrt{\frac{a-x}{a}}.$$

Therefore $\sqrt{\frac{2\mu}{a}}\ .t = a\left[\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{\frac{x-(a-x)}{a}}\right] + D.$ .... (5)

Now, when $t = o$, x =a, eq. (5) gives

$$o = a\cos^{-1} 1 + D \Rightarrow D = o.$$

Putting this value in (5), we get

$$\sqrt{\frac{2\mu}{a}}\ .t = \left[a\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{x-(a-x)}\right] \text{.................................}(6)$$

Equation (6) is the time taken by the particle to reach at P form O and OP = x.

If x= o, equation (6) becomes

$$t = \sqrt{\frac{a}{2\mu}} \quad [a\cos^{-1} o + o]$$

$$= \sqrt{\frac{a}{2\mu}} \quad \left[a\cos^{-1}\left(\cos\frac{\pi}{2}\right) + o\right]$$

$$= \sqrt{\frac{a}{2\mu}} \quad \left[a.\frac{\pi}{2}\right] = \frac{a^{3/2}}{\sqrt{2\mu}}.\frac{\pi}{2}$$

$$\Rightarrow \qquad t= \frac{a^{3/2}}{\sqrt{2\mu}}.\frac{\pi}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Therefore, time taken to reach O from A is $\frac{\pi}{2}.\frac{a^{3/2}}{\sqrt{2\mu}}$.

So, the motion of the particle is oscillatory about O and the total time T of the oscillation will be equal to 4-times the time from A to O.

i.e. $\quad T = 4 \,.\, {}^{\pi}/_{2} \,.\, \frac{a^{3/2}}{\sqrt{2\mu}}$

$$\Rightarrow T = 2\pi \,.\, \frac{a^{3/2}}{\sqrt{2\mu}}$$

Note the equation

$\frac{d^2x}{dt^2}$= -$\frac{\mu}{x^2}$ is not valid when x is negative, *i.e.* when the particle is to the left of O. For when x is negative, the right hand side is negative (μ>o), which means that the acceleration is in the direction of x decreasing, and this is not correct since to the left of O the acceleration is towards O and consequently positive.

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Calculation of Periodic time for Simple harmonic motion

If in s a simple harmonic motion, u, v, w be the velocities at distances a, b, c from a fixed point on the straight line which is not the centre of force, the periodic lime T is given by the equation.

$$\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

Proof: - The equation of simple harmonic Motion is

$$\frac{d^2x}{dt^2} = -\mu x \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \ (1)$$

Where 'x' is the distance at time 't'

$$\Rightarrow 2\,\frac{dx}{dt}.\frac{d^2x}{dt^2} = -2\,\mu x\,\frac{dx}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -2\,\mu x\,\frac{dx}{dt}$$

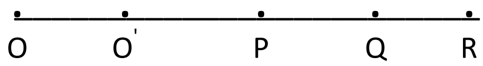
$$\Rightarrow d\left(\frac{dx}{dt}\right)^2 = -2\,\mu x\,\frac{dx}{dt}$$

$$\Rightarrow d\left(\frac{dx}{dt}\right)^2 = -2\,\mu x\,dx$$

Or . $\int d\left(\frac{dx}{dt}\right)^2 = -2\ \ \mu \int x\,dx + k,$

Where K is the constant of integration.

$$\Rightarrow d\left(\frac{dx}{dt}\right)^2 = -\mu x^2 + k \qquad \text{.............................. (2)}$$



Let y be the distance of the fixed point O' from the center of force O. Let P, Q, R be the points at distances a, b and c from O'. The velocities at these points are given to be u, v and w.

So, OP = OO' + O'P = y+a

OQ = OO' + O'Q = y+b

OR = OO' + O'R = y+c

Therefore, by using the equation (2), we can write

$$U^2 = -\mu\,(y+a)^2 + k, \qquad \text{..................... (3)}$$

$$V^2 = -\mu\,(y+b)^2 + k, \qquad \text{..................... (4)}$$

$$W^2 = -\mu\,(y+c)^2 + k, \qquad \text{..................... (5)}$$

Eq. (4) – eq. (3) gives

$$v^2 - u^2 = \mu\,[(y+a)^2 - (y+b)^2]$$

Or , $$v^2 - u^2 = \mu\,(y^2 + 2ay + a^2 - y^2 - 2by - b^2)$$

$$\Rightarrow \quad v^2 - u^2 = \mu\{2y(a - b + (a^2 - b^2)\}$$

$$\Rightarrow \quad v^2 - u^2 = \mu\{2y\,(a-b) + (a+b)(a-b)\}$$

$$\Rightarrow \quad v^2 - u^2 = \mu\,(a-b)\,(2y+a+b)$$

$$\Rightarrow \quad \frac{v^2 - u^2}{a-b} = \mu\,(2y+a+b) \qquad \text{.....................(6)}$$

Similarly , eq. (5) – eq. (4) gives

$$\frac{w^2 - v^2}{b-c} = \mu\,(2y+b+c) \qquad \text{..................... (7)}$$

From (6) and (7) , we have

$$\frac{w^2-v^2}{b-c} - \frac{v^2-u^2}{a-b} = \mu\,(2y + b + c - 2y - a - b)$$

Or, $$\frac{(w^2-v^2)(a-b)-(v^2-u^2)(b-c)}{(b-c)\,(a-b)}\ \mu(c-a)$$

Or, $$\mu(a-b)(b-c)(c-a) = (w^2 - v^2)(a-b) - (v^2 - u^2)(b-c)$$

Or,

$$\mu(b-a)(c-a)(a-b) = w^2(a-b)v^2(a-b) - v^2(-u^2)(b-c) + u^2(b-c)$$

Or, $$\mu(b-c)(c-a)(a-b) = u^2(b-c)v^2(a-b+b-c)w^2(a-b)$$

$$\Rightarrow \quad \mu(b-c)(c-a)(a-b) = u^2(b-c)v^2(c-a)w^2(a-b)$$

$$\Rightarrow \quad \mu(b-c)\ (c\text{-}a)\ (a\text{-}b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (8)$$

But, the time $\frac{2\pi}{\sqrt{\mu}}$, for a complete oscillation is called the periodic time of the motion.

Therefore,

$$T = \frac{2\pi}{\sqrt{\mu}}$$

Or, $$T^2 = \frac{4\pi^2}{\mu}$$

$$\Rightarrow \quad \mu = \frac{4\pi^2}{T^2}.$$

So, the equation (8) becomes

$$\frac{4\pi^2}{T^2}(b\text{-}c)\ (c\text{-}a)\ (a\text{-}b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

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Determination of Periodic Time for Rectilinear Motion

If a particle, whose mass in 'm', is acted upon by a force $m\mu\left(x\frac{a^4}{x^3}\right)$ towards the origin; if it starts from rest at a distance 'a', it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$.

Proof :- Let x be the distance covered from the origin O in the time 't' . Now the equation of motion can be written as

$$m.\frac{d^2x}{dt^2} = -m\mu\left(x + \frac{a^4}{x^3}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (1)$$

Here '—' sign R.H.S is taken as the force is directed towards the origin.

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\mu\left(x + \frac{a^4}{x^3}\right)$$

$$\Rightarrow \quad v.\frac{dv}{dx} = -\mu\left(x + \frac{a^4}{x^3}\right)$$

$$\Rightarrow \quad v.d\,v = -\mu x dx - \mu\, a^4 x^3 dx.$$

$$\Rightarrow \quad \int v dv = \int -\mu x dx - \mu a^4 x^3 dx.$$

$$\frac{v^2}{2} = -\mu\frac{x^2}{2} + \mu - \frac{a^4}{2x^2} + c,$$

Where c is the constant of integration. But, t=0 , x=a and

$v = o$; therefore c=o.

Therefore $\frac{v^2}{2} = -\frac{\mu}{2}\left(x^2 - \frac{a^4}{x^2}\right)$

$$\Rightarrow \quad v^2 = \mu\left(\frac{a^4}{x^2} - x^2\right)$$

$$\Rightarrow \quad v^2 = \mu\left(\frac{a^4 - x^2}{x^2}\right)$$

$$\Rightarrow \quad v^2 = \sqrt{\mu}\,.\frac{\sqrt{a^4 - x^4}}{x}$$

Here, negative sign is taken on R.H.S as t increase, x decreases. Therefore

$$\frac{dx}{dt} = \sqrt{\mu}\,.\frac{\sqrt{a^4 - x^4}}{x}$$

$$\Rightarrow \quad dt = \frac{1}{\sqrt{\mu}}\,.\frac{x}{\sqrt{(a^2)^2 - (x^2)^2}}\,.$$

For getting the time of travelling, integrating both sides under the limit form x=a to x=0,

$$\int dt = \frac{-1}{\sqrt{\mu}} \int_a^o \frac{x}{\sqrt{(a^2)^2-(x^2)^2}} dx$$

Put $x^2 = a^2\sin\theta \Rightarrow 2x dx = a^2 \cos\theta \, d\theta$

Limit for θ are o to $\frac{\pi}{2}$ as x = o to x = a.

$$\text{Therefore, } t = \frac{1}{2\sqrt{\mu}} \int_o^{\pi/2} \frac{a^2\cos\theta}{\sqrt{a^4-a^4 sin^2\,\theta}} d\theta$$

$$= \frac{1}{2\sqrt{\mu}} \int_o^{\pi/2} \frac{a^2\cos\theta}{\sqrt{a^4-a^4 sin^2\,\theta}} d\theta$$

$$= \frac{1}{2\sqrt{\mu}} \int_a^{\pi/2} \frac{\cos\theta}{\cos\theta} d\theta$$

$$= \frac{1}{2\sqrt{\mu}} \int_a^{\pi/2} d\theta$$

$$= \frac{1}{2\sqrt{\mu}} [\theta]_o^{\pi/2}$$

$$= \frac{1}{2\sqrt{\mu}} \cdot \left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow t = \frac{\pi}{4\sqrt{\mu}}.$$