Taylor's and Maclaurin's Series

(For B.Sc./B.A. Part-I, Subsidiary Course of Mathematics)

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Contents

1. Introduction

In order to understand Taylor's and Maclaurin's series, we first need to define a power series.

A series of the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots + a_n x^n + \dots\dots\dots\dots$$

where x is a variable and the coefficients $a_0, a_1, a_2, a_3, \dots\dots\dots, a_n, \dots\dots\dots$ are independent of x is called a power series in x. For example, the series

$$1 + 2x + 3x^2 + 4x^3 + \dots\dots\dots + (n+1)x^n + \dots\dots\dots\dots$$

Is a power series in x.

Taylor's and Maclaurin's series are used for representing a function in the form of a power series.

2. Taylor's Series

Statement : Let f be a function of x. If $f(x+h)$ can be expanded in a series of positive integral powers of h, then

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots\dots\dots + \frac{h^n}{n!} f^n(x) + \dots\dots\dots$$

Proof : By supposition, $f(x+h)$ can be expanded in a series of positive integral powers of h. Therefore we can suppose that

$$f(x+h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + A_4 h^4 + A_5 h^5 + \dots\dots + A_n h^n + \dots\dots \qquad \dots.(1)$$

where $A_0, A_1, A_2, A_3, A_4, A_5, \dots\dots\dots, A_n, \dots\dots$ are independent of h, whose values are to be determined.

Putting $h = 0$ in equation (1), we get

$$f(x) = A_0 \qquad \dots.(2)$$

Differentiating both sides of equation (1) w. r. t. h, we get

$$f'(x+h) = 0 + A_1(1) + A_2(2h) + A_3(3h^2) + A_4(4h^3) + A_5(5h^4) + \dots\dots + A_n(nh^{n-1}) + \dots\dots$$

i.e., $f'(x+h) = A_1 + 2A_2 h + 3A_3 h^2 + 4A_4 h^3 + 5A_5 h^4 + \dots\dots\dots + nA_n h^{n-1} + \dots\dots \qquad \dots.(3)$

Putting $h = 0$ in equation (3), we get

$$f'(x) = A_1 \qquad \dots.(4)$$

Differentiating both sides of equation (3) w. r. t. h, we get

$$f''(x+h) = 0 + 2A_2(1) + 3A_3(2h) + 4A_4(3h^2) + 5A_5(4h^3) + \dots\dots + nA_n\left((n-1)h^{n-2}\right) + \dots\dots$$

i.e.,

$$f''(x+h) = 2A_2 + (3.2)A_3h + (4.3)A_4h^2 + (5.4)A_5h^3 + ... + n(n-1)A_nh^{n-2.} + ... \quad ....(5)$$

Putting $h = 0$ in equation (5), we get

$$f''(x) = 2A_2$$

i.e., $A_2 = \dfrac{f''(x)}{2}$

or $A_2 = \dfrac{f''(x)}{2!}$ ....(6)

Differentiating both sides of equation (5) w. r. t. h, we get

$$f'''(x+h) = 0 + (3.2)A_3(1) + (4.3)A_4(2h) + (5.4)A_5(3h^2) + ... + n(n-1)A_n\left((n-2)h^{n-3}\right) + ...$$

i.e.,

$$f'''(x+h) = (3.2.1)A_3 + (4.3.2)A_4h + (5.4.3)A_5h^2 + ... + n(n-1)(n-2)A_nh^{n-3} + ... \quad ....(7)$$

Putting $h = 0$ in equation (7), we get

$$f'''(x) = (3.2.1)A_3$$

i.e., $A_3 = \dfrac{f'''(x)}{3.2.1}$

or $A_3 = \dfrac{f'''(x)}{3!}$ ....(8)

Proceeding in this way, it can be proved that

$$A_4 = \frac{f^{iv}(x)}{4!},\quad A_5 = \frac{f^{v}(x)}{5!},.........., A_n = \frac{f^{n}(x)}{n!},....... \quad ....(9)$$

Substituting the values of $A_0, A_1, A_2, A_3, A_4, A_5,......., A_n,.....$ from equations (2), (4), (6), (8), (9) In equation (1), we get

$$f(x+h) = f(x) + h\,f'(x) + h^2\left(\frac{f''(x)}{2!}\right) + h^3\left(\frac{f'''(x)}{3!}\right) + ... + h^n\left(\frac{f^{n}(x)}{n!}\right) + ...$$

i.e.,

$$\boxed{f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + ... + \frac{h^n}{n!}f^n(x) + ...}$$

3. Maclaurin's Series

Statement : Let f be a function of x. If $f(x)$ can be expanded in a series of positive integral powers of x, then

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... + \frac{x^n}{n!}f^n(0) + ...$$

Proof : By supposition, $f(x)$ can be expanded in a series of positive integral powers of x. Therefore we can suppose that

$$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ... + A_nx^n + ... \quad ....(1)$$

where $A_0, A_1, A_2, A_3, A_4, A_5, \ldots\ldots, A_n, \ldots\ldots$ are independent of x, whose values are to be determined.

Putting $x = 0$ in equation (1), we get

$$f(0) = A_0 \qquad \ldots.(2)$$

Differentiating both sides of equation (1) w. r. t. x, we get

$$f'(x) = 0 + A_1(1) + A_2(2x) + A_3(3x^2) + A_4(4x^3) + A_5(5x^4) + \ldots\ldots + A_n(nx^{n-1}) + \ldots\ldots$$

i.e., $f'(x) = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + 5A_5x^4 + \ldots\ldots + nA_nx^{n-1} + \ldots..$ ....(3)

Putting $x = 0$ in equation (3), we get

$$f'(x) = A_1 \qquad \ldots.(4)$$

Differentiating both sides of equation (3) w. r. t. x, we get

$$f''(x) = 0 + 2A_2(1) + 3A_3(2x) + 4A_4(3x^2) + 5A_5(4x^3) + \ldots.. + nA_n\left((n-1)x^{n-2}\right) + \ldots..$$

i.e., $f''(x) = 2A_2 + (3.2)A_3x + (4.3)A_4x^2 + (5.4)A_5x^3 + \ldots. + n(n-1)A_nx^{n-2} + \ldots\ldots$ ....(5)

Putting $x = 0$ in equation (5), we get

$$f''(0) = 2A_2$$

i.e., $A_2 = \dfrac{f''(0)}{2}$

or $A_2 = \dfrac{f''(0)}{2!}$ ....(6)

Differentiating both sides of equation (5) w. r. t. x, we get

$$f'''(x) = 0 + (3.2)A_3(1) + (4.3)A_4(2x) + (5.4)A_5(3x^2) + \ldots. + n(n-1)A_n\left((n-2)x^{n-3}\right) + \ldots.$$

i.e., $f'''(x) = (3.2.1)A_3 + (4.3.2)A_4x + (5.4.3)A_5x^2 + \ldots.. + n(n-1)(n-2)A_nx^{n-3} + \ldots.$ ....(7)

Putting $x = 0$ in equation (7), we get

$$f'''(0) = (3.2.1)A_3$$

i.e., $A_3 = \dfrac{f'''(0)}{3.2.1}$

or $A_3 = \dfrac{f'''(0)}{3!}$ ....(8)

Proceeding in this way, it can be proved that

$$A_4 = \frac{f^{iv}(0)}{4!},\quad A_5 = \frac{f^{v}(0)}{5!}, \ldots\ldots\ldots\ldots, A_n = \frac{f^{n}(0)}{n!}, \ldots\ldots. \qquad \ldots\ldots.(9)$$

Substituting the values of $A_0, A_1, A_2, A_3, A_4, A_5, \ldots\ldots, A_n, \ldots\ldots$ from equations (2), (4), (6), (8), (9) In equation (1), we get

$$f(x) = f(0) + x\,f'(0) + x^2\left(\frac{f''(0)}{2!}\right) + x^3\left(\frac{f'''(0)}{3!}\right) + \ldots\ldots. + x^n\left(\frac{f^{n}(0)}{n!}\right) + \ldots\ldots.$$

i.e.,

$$\boxed{f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\ldots + \frac{x^n}{n!}f^{n}(0) + \ldots\ldots..}$$

Alternative Proof of Maclaurin's Series :

Maclaurin's series can also be deduced from Taylor's series as follows :

We know that Taylor's series is

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots\dots\dots + \frac{h^n}{n!} f^n(x) + \dots\dots\dots$$

Replacing h by x and x by 0 in this series, we get

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots\dots + \frac{x^n}{n!} f^n(0) + \dots\dots\dots$$

which is Maclaurin's series.

Example 1 : Expand e^x using Maclaurin's series.

Solution : We know that Maclaurin's series is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots + \frac{x^n}{n!} f^n(0) + \dots\dots \qquad \dots\dots \ (1)$$

Let $f(x) = e^x$ ....(2)

Then $f(0) = 1.$

Differentiating both sides of equation (2) w. r. t. x successively and putting $x = 0,$ we get

$f'(x) = e^x \qquad \therefore f'(0) = 1$

$f''(x) = e^x \qquad \therefore f''(0) = 1$

$f'''(x) = e^x \qquad \therefore f'''(0) = 1$

....

....

$f^n(x) = e^x \qquad \therefore f^n(0) = 1$

Substituting the values of $f(x)$ and $f(0), f'(0), f''(0), f'''(0), \dots\dots\dots\dots, f^n(0)$ in equation (1), we get

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots + \frac{x^n}{n!} + \dots\dots$$

Example 2 : Expand $\log(1+x)$ using Maclaurin's series.

Solution : We know that Maclaurin's series is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots + \frac{x^n}{n!} f^n(0) + \dots\dots \qquad \dots\dots \ (1)$$

Let $f(x) = \log(1+x)$ ....(2)

Then $f(0) = 0.$

Differentiating both sides of equation (2) w. r. t. x successively and putting $x = 0,$ we get

$f'(x) = \frac{1}{1+x} \qquad \therefore f'(0) = 1$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad \therefore f''(0) = -1$$

$$f'''(x) = \frac{(-1)(-2)}{(1+x)^3} = \frac{(-1)^2\, 2!}{(1+x)^3} \qquad \therefore f'''(0) = 2$$

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$$f^n(x) = \frac{(-1)(-2)\ldots\ldots\ldots\ldots(-(n-1))}{(1+x)^n} = \frac{(-1)^{n-1}\,(n-1)!}{(1+x)^n} \qquad \therefore f^{\,n}(0) = (-1)^{n-1}\,(n-1)!$$

Substituting the values of $f(x)$ and $f(0), f'(0), f''(0), f'''(0), \ldots\ldots\ldots\ldots, f^n(0)$ in equation (1), we get

$$\log(1+x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \ldots\ldots + \frac{x^n}{n!}\left((-1)^{n-1}(n-1)!\right) + \ldots\ldots$$

i.e., $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots\ldots$$

Example 3 : Apply Maclaurin's series to find the expansion of $\sin x$.

Solution : We know that Maclaurin's series is

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots\ldots \qquad \ldots\ldots \; (1)$$

Let $f(x) = \sin x$ …………(2)

Then $f(0) = 0.$

Differentiating both sides of equation (2) w. r. t. x successively and putting $x = 0$, we get

$$f'(x) = \cos x \qquad \therefore f'(0) = 1$$

$$f''(x) = -\sin x \qquad \therefore f''(0) = 0$$

$$f'''(x) = -\cos x \qquad \therefore f'''(0) = -1$$

$$f^{iv}(x) = \sin x \qquad \therefore f^{iv}(0) = 0$$

$$f^{v}(x) = \cos x \qquad \therefore f^{v}(0) = 1$$

………………………

………………………

Substituting the values of $f(x)$ and $f(0), f'(0), f''(0), f'''(0), \ldots\ldots\ldots\ldots$in equation (1), we get

$$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \ldots\ldots\ldots\ldots$$

i.e., $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots\ldots$$

Example 4 : Apply Maclaurin's series to find the expansion of $\cos x.$

Solution : We know that Maclaurin's series is

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots\ldots \quad \ldots\ldots \text{ (1)}$$

Let $f(x) = \cos x$ ....(2)

Then $f(0) = 1.$

Differentiating both sides of equation (2) w. r. t. x successively and putting $x = 0,$ we get

$f'(x) = -\sin x \qquad \therefore f'(0) = 0$

$f''(x) = -\cos x \qquad \therefore f''(0) = -1$

$f'''(x) = \sin x \qquad \therefore f'''(0) = 0$

$f^{iv}(x) = \cos x \qquad \therefore f^{iv}(0) = 1$

$f^{v}(x) = -\sin x \qquad \therefore f^{v}(0) = 0$

....

....

Substituting the values of $f(x)$ and $f(0), f'(0), f''(0), f'''(0), \ldots\ldots\ldots,$ in equation (1), we get

$$\cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \ldots\ldots\ldots \ldots$$

i.e., $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\ldots \ldots$$

Example 5 : Apply Maclaurin's series to find the expansion of $\tan x.$

Solution : We know that Maclaurin's series is

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots\ldots \quad \ldots\ldots \text{ (1)}$$

Let $y = f(x) = \tan x$ ....(2)

Then $(y)_0 = f(0) = 0.$

Differentiating equation (2) w. r. t. x, we get

$y_1 = f'(x) = \sec^2 x \qquad \therefore (y_1)_0 = f'(0) = 1$

Clearly, y_1 can be written as

$y_1 = 1 + \tan^2 x = 1 + y^2$

Differentiating the above equation w. r. t. x successively and putting $x = 0,$ we get

$y_2 = 2yy_1 \qquad \therefore (y_2)_0 = f''(0) = 2(y)_0(y_1)_0 = 2(0)(1) = 0$

$$y_3 = 2\left(yy_2 + y_1^{\,2}\right) \quad \therefore (y_3)_0 = f'''(0) = 2\left[(y)_0(y_2)_0 + \{(y_1)_0\}^2\right] = 2(0+1) = 2$$

$$y_4 = 2\{(yy_3 + y_1y_2) + 2y_1y_2\} = 2(yy_3 + 3y_1y_2)$$

$$\therefore (y_4)_0 = f^{iv}(0) = 2[(y)_0(y_3)_0 + 3(y_1)_0(y_2)_0] = 2(0+0) = 0$$

$$y_5 = 2\left\{(yy_4 + y_1y_3) + 3(y_1y_3 + y_2^{\,2}\right\} = 2(yy_4 + 4y_1y_3 + 3y_2^{\,2})$$

$$\therefore (y_5)_0 = f^{v}(0) = 2\left[(y)_0(y_4)_0 + 4(y_1)_0(y_3)_0 + 3\{(y_2)_0\}^2\right] = 2[0 + 4(1)(2) + 0] = 16$$

Substituting the values of $f(x)$ and $f(0), f'(0), f''(0), f'''(0), \ldots\ldots\ldots\ldots$in equation (1), we get

$$\tan x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(16) + \ldots\ldots\ldots\ldots$$

i.e., $$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots\ldots\ldots$$

Example 6 : Expand $\log\sin(x+h)$ in ascending powers of h as far as the term involving h^3.

Solution : We know that Taylor's series is

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots \qquad (1)$$

Let $f(x) = \log\sin x$ ....(2)

Then $f(x+h)) = \log\sin(x+h)$

Differentiating both sides of equation (2) w. r. t. x successively, we get

$$f'(x) = \frac{1}{\sin x}.\cos x = \cot x$$

$$f''(x) = -\operatorname{cosec}^2 x$$

$$f'''(x) = -(2\operatorname{cosec}x)(-\operatorname{cosec}x\cot x) = 2\operatorname{cosec}^2 x\cot x$$

....

....

Substituting the values of $f(x+h)$ and $f(x), f'(x), f''(x), f'''(x), \ldots\ldots\ldots\ldots$, in equation (1), we get

$$\log\sin(x+h) = \log\sin x + \frac{h}{1!}(\cot x) + \frac{h^2}{2!}(-\operatorname{cosec}^2 x) + \frac{h^3}{3!}(2\operatorname{cosec}^2 x\cot x) + \ldots\ldots$$

i.e., $$\log\sin(x+h) = \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\operatorname{cosec}^2 x\cot x + \ldots\ldots\ldots$$

Example 7 : Expand $\log\cos x$ using Maclaurin's series.

Solution : We know that Maclaurin's series is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \ldots\ldots \quad \ldots\ldots \text{(1)}$$

Let $y = f(x) = \log\cos x$ ....(2)

Then $(y)_0 = f(0) = 0.$

Differentiating equation (2) w. r. t. $x,$ we get

$$y_1 = f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x \qquad \therefore (y_1)_0 = f'(0) = 0$$

Differentiating y_1 w. r. t. $x,$ we get

$$y_2 = -\sec^2 x \qquad \therefore (y_2)_0 = f''(0) = -1$$

Clearly, y_2 can be written as

$$y_2 = -\left(1 + (-\tan x)^2\right) = -\left(1 + y_1^{\,2}\right)$$

Differentiating the above equation w. r. t. x successively and putting $x = 0,$ we get

$$y_3 = -2y_1y_2 \qquad \therefore (y_3)_0 = f'''(0) = -2(y_1)_0(y_2)_0 = -2(0)(-1) = 0$$

$$y_4 = -2\left(y_1y_3 + y_2^{\,2}\right) \quad \therefore (y_4)_0 = f^{iv}(0) = -2\left[(y_1)_0(y_3)_0 + \{(y_2)_0\}^2\right] = -2(0+1) = -2$$

$$y_5 = -2\{(y_1y_4 + y_2y_3) + 2y_2y_3\} = -2(y_1y_4 + 3y_2y_3)$$

$$\therefore (y_5)_0 = f^{\text{v}}(0) = -2\left[(y_1)_0(y_4)_0 + 3(y_2)_0(y_3)_0\right] = -2(0+0) = 0$$

$$y_6 = -2\left\{(y_1y_5 + y_2y_4) + 3(y_2y_4 + y_3^{\,2}\right\} = -2(y_1y_5 + 4y_2y_4 + 3y_3^{\,2})$$

$$\therefore (y_6)_0 = f^{\text{vi}}(0) = -2\left[(y_1)_0(y_5)_0 + 4(y_2)_0(y_4)_0 + 3\{(y_3)_0\}^2\right] = -2\left[0 + 4(-1)(-2) + 0\right] = -16$$

Substituting the values of $f(x)$ and $f(0), f'(0), f''(0), f'''(0),$............in equation (1), we get

$$\log\cos x = 0 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-2) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-16) + \ldots\ldots\ldots\ \ldots.$$

i.e., $$\log\cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \ldots\ldots\ldots\ \ldots.$$

Example 8 : Expand $\log(1+\sin x)$ in ascending powers of x using Maclaurin's series up to x^4.

Solution : We know that Maclaurin's series is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots\ldots \quad \ldots\ldots \text{(1)}$$

Let $y = f(x) = \log(1+\sin x)$ ....(2)

Then $(y)_0 = f(0) = 0.$

From equation (2), we have

$$e^{y} = 1+\sin x \qquad .....(3)$$

Differentiating both sides of equation (3) w. r. t. $x,$ we get

$$e^{y} y_1 = \cos x \qquad ....(4)$$

Therefore $e^{(y)_0}(y_1)_0 = 1$

This $\Rightarrow e^{0}(y_1)_0 = 1$

$\Rightarrow 1(y_1)_0 = 1$

$\Rightarrow (y_1)_0 = 1,$ i.e., $f\,'(0) = 1.$

Differentiating equation (4) w. r. t. x, we get

$$e^{y} y_2 + e^{y} {y_1}^2 = -\sin x \qquad ....(5)$$

$\therefore \quad e^{(y)_0}(y_2)_0 + e^{(y)_0}\{(y_1)_0\}^2 = 0$

This $\Rightarrow e^{0}(y_2)_0 + e^{0}(1) = 0$

$\Rightarrow 1(y_2)_0 + 1(1) = 0$

$\Rightarrow (y_2)_0 + 1 = 0$

$\Rightarrow (y_2)_0 = -1,$ i.e., $f\,''(0) = -1.$

Differentiating equation (5) w. r. t. x, we get

$$(e^{y} y_3 + e^{y} y_1 y_2) + (2y_1 y_2 e^{y} + e^{y} {y_1}^3) = -\cos x$$

i.e., $e^{y} y_3 + 3e^{y} y_1 y_2 + e^{y} {y_1}^3 = -\cos x \qquad .....(6)$

$\therefore \quad e^{(y)_0}(y_3)_0 + 3e^{(y)_0}(y_1)_0 (y_2)_0 + e^{(y)_0}\{(y_1)_0\}^3 = -1$

This $\Rightarrow e^{0}(y_3)_0 + 3e^{0}(1)(-1) + e^{0}(1) = -1$

$\Rightarrow 1(y_3)_0 + 3(1)(1)(-1) + 1(1) = -1$

$\Rightarrow (y_3)_0 - 3 + 1 = -1$

$\Rightarrow (y_3)_0 - 2 = -1$

$\Rightarrow (y_3)_0 = 2 - 1 = 1,$ i.e., $f\,'''(0) = 1.$

Again, differentiating equation (6) w. r. t. x, we get

$$(e^{y} y_4 + e^{y} y_1 y_3) + 3(e^{y} y_1 y_3 + e^{y} {y_2}^2 + e^{y} {y_1}^2 y_2) + (3{y_1}^2 y_2 e^{y} + e^{y} {y_1}^4) = \sin x$$

i.e., $e^{y} y_4 + 4e^{y} y_1 y_3 + 3e^{y} {y_2}^2 + 6e^{y} {y_1}^2 y_2 + e^{y} {y_1}^4 = \sin x$

$\therefore \quad e^{(y)_0}(y_4)_0 + 4e^{(y)_0}(y_1)_0 (y_3)_0 + 3e^{(y)_0}\{(y_2)_0\}^2 + 6e^{(y)_0}\{(y_1)_0\}^2 (y_2)_0 + e^{(y)_0}\{(y_1)_0\}^4 = 0$

This $\Rightarrow e^{0}(y_4)_0 + 4e^{0}(1)(1) + 3e^{0}(1) + 6e^{0}(1)(-1) + e^{0}(1) = 0$

$$\Rightarrow 1(y_4)_0 + 4(1)(1)(1) + 3(1)(1) + 6(1)(1)(-1) + 1(1) = 0$$

$$\Rightarrow (y_4)_0 + 4 + 3 - 6 + 1 = 0$$

$$\Rightarrow (y_4)_0 + 2 = 0$$

$$\Rightarrow (y_4)_0 = -2, \quad \text{i.e.,} \quad f^{iv}(0) = -2.$$

Substituting the values of $f(x)$ and $f(0), f'(0), f''(0), f'''(0),$..........in equation (1), we get

$$\log(1+\sin x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(-2) + \dots\dots\dots\dots$$

i.e., $$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots\dots\dots\dots$$

Example 9 : Obtain the expansion of $e^{ax}\cos bx$ in ascending powers of x.

Solution : We know that Maclaurin's series is

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \dots\dots \qquad \dots\dots (1)$$

Let $y = f(x) = e^{ax}\cos bx$ ....(2)

Then $(y)_0 = f(0) = 1.$

Differentiating both sides of equation (2) w. r. t. x, we get

$$y_1 = f'(x) = ae^{ax}\cos bx + e^{ax}(-b\sin bx) = ay - be^{ax}\sin bx \qquad \dots\dots(3)$$

From equation (3), we have

$$(y_1)_0 = f'(0) = a(y)_0 - 0 = a(1) - 0 = a.$$

Differentiating equation (3) w. r. t. x, we get

$$y_2 = a\,y_1 - b\,(ae^{ax}\sin bx + be^{ax}\cos bx)$$

$$= a\,y_1 - a\,(be^{ax}\sin bx) - b^2e^{ax}\cos bx$$

$$= a\,y_1 - a\,(a\,y - y_1) - b^2 y \qquad \text{[using equations (2) and (3)]}$$

i.e., $$y_2 = 2a\,y_1 - (a^2+b^2)y \qquad \dots\dots\dots(4)$$

From equation (4), We have

$$(y_2)_0 = f''(0) = 2a(y_1)_0 - (a^2+b^2)(y)_0 = 2a(a) - (a^2+b^2)(1) = a^2 - b^2.$$

Differentiating both sides of equation (4) w. r. t. x, we get

$$y_3 = 2a\,y_2 - (a^2+b^2)y_1 \qquad \dots\dots\dots(5)$$

From equation (5), We have

$$(y_3)_0 = f'''(0) = 2a(y_2)_0 - (a^2+b^2)(y_1)_0 = 2a(a^2-b^2) - (a^2+b^2)(a) = a^3 - 3ab^2.$$

Differentiating both sides of equation (5) w. r. t. x, we get

$$y_4 = 2a\,y_3 - (a^2+b^2)y_2 \qquad \dots\dots\dots(6)$$

From equation (6), We have

$$(y_4)_0 = f^{iv}(0) = 2a(y_3)_0 - (a^2+b^2)(y_2)_0$$
$$= 2a(a^3 - 3ab^2) - (a^2+b^2)(a^2-b^2)$$
$$= (2a^4 - 6a^2b^2) - (a^4 - b^4)$$
$$= a^4 - 6a^2b^2 + b^4$$

Substituting the values of $f(x)$ and $f(0), f'(0), f''(0), f'''(0),$............in equation (1), we get

$$e^{ax}\cos bx = 1 + \frac{x}{1!}(a) + \frac{x^2}{2!}(a^2-b^2) + \frac{x^3}{3!}(a^3-3ab^2) + \frac{x^4}{4!}(a^4-6a^2b^2+b^4) + \ldots\ldots$$

i.e.,
$$e^{ax}\cos bx = 1 + ax + \frac{x^2}{2}(a^2-b^2) + \frac{x^3}{6}(a^3-3ab^2) + \frac{x^4}{24}(a^4-6a^2b^2+b^4) + \ldots\ldots$$

Example 10 : Expand $\tan^{-1} x$.

Solution : We know that Maclaurin's series is

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots \quad \ldots\ldots. \ (1)$$

Let $y = f(x) = \tan^{-1} x$ ....(2)

Then $(y)_0 = f(0) = 0.$

Differentiating equation (2) w. r. t. $x,$ we get

$$y_1 = f'(x) = \frac{1}{1+x^2} \quad \ldots..(3)$$

From equation (3), we have

$$(y_1)_0 = f'(0) = 1.$$

Also, from equation (3), we have

$$(1+x^2)y_1 = 1 \quad \ldots..(4)$$

Differentiating both sides of equation (4) w. r. t. $x,$ we get

$$(1+x^2)y_2 + 2xy_1 = 0 \quad \ldots...(5)$$

From equation (5), we have

$(y_2)_0 = 0,$ i.e., $f''(0) = 0.$

Differentiating both sides of equation (5) n times w. r. t. x and using Leibnitz's Theorem, we get

$$[y_{n+2}(1+x^2) + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n(2)] + 2[y_{n+1}x + {}^nC_1 y_n(1)] = 0$$

(by Leibnitz's Theorem, $(uv)_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \ldots\ldots\ldots + {}^nC_n u v_n$)

This $\Rightarrow \left[y_{n+2}(1+x^2) + n\, y_{n+1}(2x) + \frac{n(n-1)}{2} y_n(2) \right] + 2\left[y_{n+1}x + n\, y_n \right] = 0$

$$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0 \quad \ldots.(6)$$

Putting $x = 0$ in equation (6), we get

$$(y_{n+2})_0 = -n\,(n+1)(y_n)_0 \qquad .....(7)$$

We have already proved that

$$\left.\begin{aligned} f(0) &= (y)_0 = 0 \\ f'(0) &= (y_1)_0 = 1 \\ f''(0) &= (y_2)_0 = 0 \end{aligned}\right\} \qquad ....(8)$$

Putting $n = 1, 2, 3, 4, ...$ successively in equation (7) and using equation (8), we get

$$\left.\begin{aligned} f'''(0) &= (y_3)_0 = -2 \\ f^{iv}(0) &= (y_4)_0 = 0 \\ f^{v}(0) &= (y_5)_0 = 24 \\ f^{vi}(0) &= (y_6)_0 = 0 \end{aligned}\right\} \qquad ......(9)$$

Substituting the values of $f(x)$ and $f(0), f'(0), f''(0), f'''(0), ...$ in equation (1), we get

$$\tan^{-1} x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(24) + \frac{x^6}{6!}(0) + ...$$

i.e., $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - ....$$

Exercises

1. Find the expansion of $(1+x)^n$, where n is either negative or fractional.
2. Apply Maclaurin's series to find the expansion of $\sec x$.
3. Expand $\log(1+\tan x)$ in ascending powers of x using Maclaurin's series.
4. Expand $\log(1+e^x)$ in ascending powers of x using Maclaurin's series up to the term of x^3.
5. Expand $e^{\sin x}$ in ascending powers of x using Maclaurin's series up to the term involving x^4.
6. Expand $\sin^{-1} x$.

Answers

1. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$
2. $\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + ...$
3. $\log(1+\tan x) = x - \frac{x^2}{2} + \frac{2}{3}x^3 - ....$
4. $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + ...$
5. $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + ...$
6. $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + ...$