Successive Differentiation

(For B.Sc./B.A. Part-I, Hons. And Subsidiary Courses of Mathematics)

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Contents

1. Definitions and Notations

Successive differentiation is the differentiation of a function successively to derive its higher order derivatives.

If $y = f(x)$ be a function of x, then the derivative (or differential coefficient) of y w. r. t. x is denoted by $\frac{dy}{dx}$ or Dy or $f'(x)$ or y_1 and this is called the first derivative of y w. r. t. x, where $D \equiv \frac{d}{dx}$.

If $\frac{dy}{dx}$ can be differentiated once again, i.e., if $y = f(x)$ is derivable twice w. r. t. x, then the derivative of $\frac{dy}{dx}$ w. r. t. x is denoted by $\frac{d^2y}{dx^2}$ or D^2y or $f''(x)$ or y_2 and this is called the second derivative of y w. r. t. x.

Similarly, if $\frac{d^2y}{dx^2}$ can be differentiated once again, i.e., if $y = f(x)$ is derivable thrice w. r. t. x, then the derivative of $\frac{d^2y}{dx^2}$ w. r. t. x is denoted by $\frac{d^3y}{dx^3}$ or D^3y or $f'''(x)$ or y_3 and this is called the third derivative of y w. r. t. x.

In a similar manner, we can find the fourth derivative, fifth derivative and, in general, the nth derivative of y w. r. t. x by differentiating successively the given function y w. r. t. x four times, five times and n times.

Following notations are generally used for the successive derivatives of y w. r. t. x :

	First derivative	Second derivative	Third derivative	….	nth derivative
or	y_1	y_2	y_3	….	y_n
or	$f'(x)$	$f''(x)$	$f'''(x)$	….	$f^n(x)$
or	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	….	$\frac{d^ny}{dx^n}$
or	Dy	D^2y	D^3y	….	D^ny

Example 1 : If $\sqrt{x+y}+\sqrt{y-x}=c$, prove that $y_2=\frac{2}{c^2}$.

Solution : Given that $\sqrt{x+y}+\sqrt{y-x}=c$

Squaring both sides, we get

$$x+y+y-x+2\sqrt{y^2-x^2}=c^2$$

This $\Rightarrow 2\sqrt{y^2-x^2}=c^2-2y$

Again, squaring both sides, we get

$$4\left(y^2-x^2\right)=c^4-4c^2y+4y^2$$

This $\Rightarrow 4x^2-4c^2y+c^4=0$

Differentiating both sides w. r. t. x, we get

$$8x-4c^2y_1=0$$

This $\Rightarrow 2x-c^2y_1=0$

Again, differentiating both sides w. r. t. x, we get

$$2-c^2y_2=0$$

This $\Rightarrow y_2=\frac{2}{c^2}$.

Example 2 : If $y=a\cos(\log x)+b\sin(\log x)$, prove that $x^2y_2+xy_1+y=0$.

Solution : Given that $y=a\cos(\log x)+b\sin(\log x)$.

Differentiating both sides w. r. t. x, we get

$$y_1=-a\sin(\log x).\frac{1}{x}+b\cos(\log x).\frac{1}{x}$$

This $\Rightarrow xy_1=-a\sin(\log x)+b\cos(\log x)$

Again, differentiating both sides w. r. t. x, we get

$$xy_2+y_1=-a\cos(\log x).\frac{1}{x}-b\sin(\log x).\frac{1}{x}$$

This $\Rightarrow x^2y_2+xy_1=-a\cos(\log x)-b\sin(\log x)$

$$=-\left(a\cos(\log x)+b\sin(\log x)\right)$$

$$=-y \quad \left(\because y=a\cos(\log x)+b\sin(\log x) \text{ given}\right)$$

This $\Rightarrow x^2y_2+xy_1+y=0$.

Example 3 : If $y = e^{a\sin^{-1}x}$, prove that $(1-x^2)y_2 - xy_1 = a^2 y.$

Solution : Given that $y = e^{a\sin^{-1}x}$.

Differentiating both sides w. r. t. x, we get

$$y_1 = e^{a\sin^{-1}x}\left(a\frac{1}{\sqrt{1-x^2}}\right)$$

This$\Rightarrow y_1 = \frac{ay}{\sqrt{1-x^2}}$ $\quad \left(\because y = e^{a\sin^{-1}x} \text{ given}\right)$

$$\Rightarrow y_1\sqrt{1-x^2} = ay$$

Squaring both sides, we get

$$y_1^{\,2}(1-x^2) = a^2 y^2$$

Again, differentiating both sides w. r. t. x, we get

$$2y_1 y_2(1-x^2) + y_1^{\,2}(-2x) = a^2\,(2yy_1)$$

This$\Rightarrow$ $2y_1\left[(1-x^2)\,y_2 - xy_1\right] = 2y_1(a^2 y)$

$$\Rightarrow (1-x^2)\,y_2 - xy_1 = a^2 y.$$

Example 4 : If $x = \cosh\left(\frac{\log y}{m}\right)$, prove that $(x^2-1)y_2 + xy_1 - m^2 y = 0.$

Solution : Given that $x = \cosh\left(\frac{\log y}{m}\right)$.

This$\Rightarrow$ $\cosh^{-1} x = \frac{\log y}{m}$

Differentiating both sides w. r. t. x, we get

$$\frac{1}{\sqrt{x^2-1}} = \frac{1}{my}y_1$$

This$\Rightarrow$ $y_1\sqrt{x^2-1} = my$

Squaring both sides, we get

$$y_1^{\,2}(x^2-1) = m^2 y^2$$

Again, differentiating both sides w. r. t. x, we get

$$2y_1 y_2(x^2-1) + y_1^{\,2}(2\,x) = m^2\,(2yy_1)$$

This$\Rightarrow$ $2y_1\left[(x^2-1)\,y_2 + xy_1\right] = 2y_1(m^2 y)$

$$\Rightarrow (x^2-1)\,y_2 + xy_1 = m^2 y.$$

2. Derivation of n[th] differential coefficient of Some Standard Functions

1. n[th] differential coefficient of x^m, where m is a positive integer $\geq n$

Let $y = x^m$

Differentiating both sides w. r. t. x successively, we get

$$y_1 = m\,x^{m-1}$$

$$y_2 = m(m-1)x^{m-2}$$

$$y_3 = m(m-1)(m-2)x^{m-3}$$

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Now we suppose that

$$y_n = m(m-1)(m-2)(m-3)..............(m-(n-1))x^{m-n} \qquad ....(1)$$

Differentiating both sides of equation (1) w. r. t. x, we get

$$y_{n+1} = m(m-1)(m-2)(m-3)..............(m-(n-1))(m-n)x^{m-(n+1)}$$

Clearly, y_{n+1} is of the same form as y_n, given by equation (1). This $\Rightarrow$ If equation (1) is true for a particular value of n, then equation (1) is true for next higher value of n also.

But we have already proved that equation (1) is true for n = 1, 2, 3.

Since equation (1) is true for n = 3, it is true for n = 4 also. Similarly, since equation (1) is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, equation (1) is true for every positive integer n.

From equation (1), we can write y_n as follows:

$$y_n = \frac{m(m-1)(m-2)(m-3)..............(m-(n-1))(m-n)(m-(n+1))..........3.2.1}{(m-n)(m-(n+1))..........3.2.1}x^{m-n}$$

i.e., $$\boxed{y_n = \frac{m\,!}{(m-n)\,!}x^{m-n}}$$

Corollary : If $y = x^n$, then $y_n = n!$

2. n[th] differential coefficient of $(ax+b)^m$, where m is a positive integer $\geq n$

Let $y = (ax+b)^m$

Differentiating both sides w. r. t. x successively, we get

$$y_1 = m\,a(ax+b)^{m-1}$$

$$y_2 = m(m-1)a^2(ax+b)^{m-2}$$

$$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$$

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Now we suppose that

$$y_n = m(m-1)(m-2)(m-2)..............(m-(n-1))a^n(ax+b)^{m-n} \qquad ....(2)$$

Differentiating both sides of equation (2) w. r. t. x, we get

$$y_{n+1} = m(m-1)(m-2)(m-3)\ldots\ldots\ldots\ldots(m-(n-1))(m-n)a^{n+1}(ax+b)^{m-(n+1)}$$

Clearly, y_{n+1} is of the same form as y_n, given by equation (2). This $\Rightarrow$ If equation (2) is true for a particular value of n, then equation (2) is true for next higher value of n also.

But we have already proved that equation (2) is true for n = 1, 2, 3.

Since equation (2) is true for n = 3, it is true for n = 4 also. Similarly, since equation (2) is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, equation (2) is true for every positive integer n.

From equation (2), we can write y_n as follows:

$$y_n = \frac{m(m-1)(m-2)(m-2)\ldots\ldots\ldots(m-(n-1))(m-n)(m-n-1)\ldots\ldots 3.2.1}{(m-n)(m-n-1)\ldots\ldots\ldots 3.2.1}a^n(ax+b)^{m-n}$$

i.e., $$\boxed{y_n = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}}$$

3. n^th^ differential coefficient of $\frac{1}{ax+b}$

Let $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$

Differentiating y w. r. t. x successively, we get

$$y_1 = (-1)a(ax+b)^{-2}$$

$$y_2 = (-1)(-2)a^2(ax+b)^{-3}$$

$$y_3 = (-1)(-2)(-3)a^3(ax+b)^{-4}$$

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Now we suppose that

$$y_n = (-1)(-2)(-3)\ldots\ldots\ldots\ldots(-n)a^n(ax+b)^{-(n+1)} \qquad \ldots\ldots(3)$$

Differentiating both sides of equation (3) w. r. t. x, we get

$$y_{n+1} = (-1)(-2)(-3)\ldots\ldots\ldots\ldots(-n)(-(n+1))a^{n+1}(ax+b)^{-(n+2)}$$

Clearly, y_{n+1} is of the same form as y_n, given by equation (3). This $\Rightarrow$ If equation (3) is true for a particular value of n, then equation (3) is true for next higher value of n also.

But we have already proved that equation (3) is true for n = 1, 2, 3.

Since equation (3) is true for n = 3, it is true for n = 4 also. Similarly, since equation (3) is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, equation (3) is true for every positive integer n.

From equation (3), we can write y_n as follows:

$$y_n = (-1)^n n!a^n(ax+b)^{-(n+1)}$$

i.e., $$\boxed{y_n = \frac{(-1)^n a^n n!}{(ax+b)^{(n+1)}}}$$

4. n[th] **differential coefficient of** $\dfrac{1}{(ax+b)^2}$

Let $y = \dfrac{1}{(ax+b)^2} = (ax+b)^{-2}$

Differentiating both sides w. r. t. x successively, we get

$$y_1 = (-2)a(ax+b)^{-3}$$

$$y_2 = (-2)(-3)\,a^2\,(ax+b)^{-4}$$

$$y_3 = (-2)(-3)(-4)a^3\,(ax+b)^{-5}$$

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Now we suppose that

$$y_n = (-2)(-3)(-4)\ldots\ldots\ldots\ldots(-(n+1))a^n\,(ax+b)^{-(n+2)} \qquad \ldots\ldots.(4)$$

Differentiating both sides of equation (4) w. r. t. x, we get

$$y_{n+1} = (-2)(-3)(-4)\ldots\ldots\ldots\ldots(-(n+1))\,(-(n+2))a^{n+1}\,(ax+b)^{-(n+3))}$$

Clearly, y_{n+1} is of the same form as y_n, given by equation (4). This $\Rightarrow$ If equation (4) is true for a particular value of n, then equation (4) is true for next higher value of n also.

But we have already proved that equation (4) is true for n = 1, 2, 3.

Since equation (4) is true for n = 3, it is true for n = 4 also. Similarly, since equation (4) is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, equation (4) is true for every positive integer n.

From equation (4), we can write y_n as follows:

$$y_n = (-1)^n\,(n+1)!\,a^n\,(ax+b)^{-(n+2)}$$

i.e., $$\boxed{y_n = \dfrac{(-1)^n\,a^n\,(n+1)!}{(ax+b)^{(n+2)}}}$$

5. n[th] **differential coefficient of log $(ax+b)$**

Let $y = \log\,(ax+b)$

Differentiating both sides w. r. t. x successively, we get

$$y_1 = \frac{a}{ax+b} = a\,(ax+b)^{-1}$$

$$y_2 = (-1)\,a^2\,(ax+b)^{-2}$$

$$y_3 = (-1)(-2)a^3\,(ax+b)^{-3}$$

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Now we suppose that

$$y_n = (-1)(-2)(-3)..............(-(n-1))a^n(ax+b)^{-n} \qquad ....(5)$$

Differentiating both sides of equation (5) w. r. t. x, we get

$$y_{n+1} = (-1)(-2)(-3)..............(-(n-1))(-n)a^{n+1}(ax+b)^{-(n+1)}$$

Clearly, y_{n+1} is of the same form as y_n, given by equation (5). This $\Rightarrow$ If equation (5) is true for a particular value of n, then equation (5) is true for next higher value of n also.

But we have already proved that equation (5) is true for n = 1, 2, 3.

Since equation (5) is true for n = 3, it is true for n = 4 also. Similarly, since equation (5) is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, equation (5) is true for every positive integer n.

From equation (5), we can write y_n as follows:

$$y_n = (-1)^{n-1}(n-1)!\,a^n(ax+b)^{-n}$$

i.e., $$\boxed{y_n = \frac{(-1)^{n-1}a^n\,(n-1)!}{(ax+b)^n}}$$

6. n^th differential coefficient of e^{mx}

Let $y = e^{mx}$

Differentiating both sides w. r. t. x successively, we get

$$y_1 = me^{mx}$$

$$y_2 = m^2e^{mx}$$

$$y_3 = m^3e^{mx}$$

....

....

Now we suppose that

$$y_n = m^n e^{mx} \qquad ....(6)$$

Differentiating both sides of equation (6) w. r. t. x, we get

$$y_{n+1} = m^n(me^{mx}) \qquad \text{i.e.,} \quad y_{n+1} = m^{n+1}e^{mx}$$

Clearly, y_{n+1} is of the same form as y_n, given by equation (6). This $\Rightarrow$ If equation (6) is true for a particular value of n, then equation (6) is true for next higher value of n also.

But we have already proved that equation (6) is true for n = 1, 2, 3.

Since equation (6) is true for n = 3, it is true for n = 4 also. Similarly, since equation (6) is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, equation (6) is true for every positive integer n,

i.e., $$\boxed{y_n = m^n e^{mx}}$$

7. n[th] differential coefficient of a^{mx}

Let $y = a^{mx}$

Then $y = e^{mx\log a} \qquad \left(\because a^{mx} = e^{mx\log a}\right)$

Differentiating both sides w. r. t. x successively, we get

$$y_1 = m\log a\left(e^{mx\log a}\right)$$
$$y_2 = m\log a\left\{m\log a\left(e^{mx\log a}\right)\right\} = (m\log a)^2\left(e^{mx\log a}\right)$$
$$y_3 = (m\log a)^2\left\{m\log a\left(e^{mx\log a}\right)\right\} == (m\log a)^3\left(e^{mx\log a}\right)$$

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Now we suppose that

$$y_n = (m\log a)^n\left(e^{mx\log a}\right) \qquad \text{……..(6)}$$

Differentiating both sides of equation (6) w. r. t. x, we get

$$y_{n+1} = (m\log a)^n\left\{m\log a\left(e^{mx\log a}\right)\right\} == (m\log a)^{n+1}\left(e^{mx\log a}\right)$$

Clearly, y_{n+1} is of the same form as y_n, given by equation (6). This $\Rightarrow$ If equation (6) is true for a particular value of n, then equation (6) is true for next higher value of n also.

But we have already proved that equation (6) is true for n = 1, 2, 3.

Since equation (6) is true for n = 3, it is true for n = 4 also. Similarly, since equation (6) is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, equation (6) is true for every positive integer n,

$y_n = (m\log a)^n\left(e^{mx\log a}\right)$

i.e., $y_n = (m\log a)^n\, a^{mx} \qquad \left(\because a^{mx} = e^{mx\log a}\right)$

i.e., $\boxed{y_n = m^n(\log a)^n\, a^{mx}}$

8. n[th] differential coefficient of sin $(ax+b)$

Let $y = \sin(ax+b)$

Differentiating both sides w. r. t. x successively, we get

$$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$$
$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+2\frac{\pi}{2}\right)$$
$$y_3 = a^3\cos\left(ax+b+2\frac{\pi}{2}\right) = a^3\sin\left(ax+b+2\frac{\pi}{2}+\frac{\pi}{2}\right) = a^3\sin\left(ax+b+3\frac{\pi}{2}\right)$$

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Now we suppose that

$$y_n = a^n \sin\left(ax + b + n\frac{\pi}{2}\right) \qquad \text{..........(7)}$$

Differentiating both sides of equation (7) w. r. t. x, we get

$$y_{n+1} = a^{n+1} \cos\left(ax + b + n\frac{\pi}{2}\right) = a^{n+1} \sin\left(ax + b + n\frac{\pi}{2} + \frac{\pi}{2}\right)$$

i.e., $$y_{n+1} = a^{n+1} \sin\left(ax + b + (n+1)\frac{\pi}{2}\right)$$

Clearly, y_{n+1} is of the same form as y_n, given by equation (7). This $\Rightarrow$ If equation (7) is true for a particular value of n, then equation (7) is true for next higher value of n also.

But we have already proved that equation (7) is true for n = 1, 2, 3.

Since equation (7) is true for n = 3, it is true for n = 4 also. Similarly, since equation (7) is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, equation (7) is true for every positive integer n,

i.e., $$\boxed{y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)}$$

9. n^th^ differential coefficient of $\cos(ax + b)$

Let $y = \cos(ax + b)$

Differentiating both sides w. r. t. x successively, we get

$$y_1 = -a \sin(ax + b) = a \cos\left(ax + b + \frac{\pi}{2}\right)$$

$$y_2 = -a^2 \sin\left(ax + b + \frac{\pi}{2}\right) = a^2 \cos\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right) = a^2 \cos\left(ax + b + 2\frac{\pi}{2}\right)$$

$$y_3 = -a^3 \sin\left(ax + b + 2\frac{\pi}{2}\right) = a^3 \cos\left(ax + b + 2\frac{\pi}{2} + \frac{\pi}{2}\right) = a^3 \cos\left(ax + b + 3\frac{\pi}{2}\right)$$

....

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Now we suppose that

$$y_n = a^n \cos\left(ax + b + n\frac{\pi}{2}\right) \qquad \text{..........(8)}$$

Differentiating both sides of equation (8) w. r. t. x, we get

$$y_{n+1} = -a^{n+1} \sin\left(ax + b + n\frac{\pi}{2}\right) = a^{n+1} \cos\left(ax + b + n\frac{\pi}{2} + \frac{\pi}{2}\right)$$

i.e., $$y_{n+1} = a^{n+1} \cos\left(ax + b + (n+1)\frac{\pi}{2}\right)$$

Clearly, y_{n+1} is of the same form as y_n, given by equation (8). This $\Rightarrow$ If equation (8) is true for a particular value of n, then equation (8) is true for next higher value of n also.

But we have already proved that equation (8) is true for n = 1, 2, 3.

Since equation (8) is true for n = 3, it is true for n = 4 also. Similarly, since equation (8) is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, equation (8) is true for every positive integer n,

i.e., $$\boxed{y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)}$$

10. n[th] differential coefficient of $e^{ax}\sin(bx+c)$

Let $y = e^{ax}\sin(bx+c)$

Differentiating both sides w. r. t. x , we get

$$y_1 = a\,e^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$$

Put $a = r\cos\theta$ and $b = r\sin\theta$ so that $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\dfrac{b}{a}$

Then we have

$$y_1 = r\cos\theta\, e^{ax}\sin(bx+c) + r\sin\theta\, e^{ax}\cos(bx+c)$$

$$= r\,e^{ax}\left(\sin(bx+c)\cos\theta + \cos(bx+c)\sin\theta\right)$$

$$= r\,e^{ax}\sin(bx+c+\theta)$$

Similarly, differentiating y w. r. t. x twice, thrice, ………, we get

$$y_2 = r^2\, e^{ax}\sin(bx+c+2\theta)$$

$$y_3 = r^3\, e^{ax}\sin(bx+c+3\theta)$$

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Now we suppose that

$$y_n = r^n\, e^{ax}\sin(bx+c+n\theta) \qquad ....(9)$$

Differentiating both sides of equation (9) w. r. t. x, we get

$$y_{n+1} = r^n\left[ae^{ax}\sin(bx+c+n\theta) + be^{ax}\cos(bx+c+n\theta)\right]$$

$$= r^n e^{ax}\left[a\sin(bx+c+n\theta) + b\cos(bx+c+n\theta)\right]$$

$$= r^n e^{ax}\left[r\cos\theta\sin(bx+c+n\theta) + r\sin\theta\cos(bx+c+n\theta)\right]$$

(using $a = r\cos\theta,\ b = \sin\theta$)

$$= r^{n+1}e^{ax}\left[\sin(bx+c+n\theta)\cos\theta + \cos(bx+c+n\theta)\sin\theta\right]$$

$$= r^{n+1}e^{ax}\sin(bx+c+n\theta+\theta)$$

$$= r^{n+1}e^{ax}\sin\left(bx+c+(n+1)\theta\right)$$

Clearly, y_{n+1} is of the same form as y_n, given by equation (9). This $\Rightarrow$ If equation (9) is true for a particular value of n, then equation (9) is true for next higher value of n also.

But we have already proved that equation (9) is true for n = 1, 2, 3.

Since equation (9) is true for n = 3, it is true for n = 4 also. Similarly, since equation (9) is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, equation (9) is true for every positive integer n,

i.e.,

$$\boxed{y_n = r^n\, e^{ax}\sin(bx+c+n\theta), \quad \text{where } r = \sqrt{a^2+b^2} \ \text{ and } \ \theta = \tan^{-1}\frac{b}{a}}$$

11. n^{th} differential coefficient of $e^{ax}\cos(bx+c)$

Let $y = e^{ax}\cos(bx+c)$

Differentiating both sides w. r. t. x, we get

$$y_1 = a\,e^{ax}\cos(bx+c) - be^{ax}\sin(bx+c)$$

Put $a = r\cos\theta$ and $b = r\sin\theta$ so that $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\dfrac{b}{a}$

Then we have

$$y_1 = r\cos\theta\, e^{ax}\cos(bx+c) - r\sin\theta\, e^{ax}\sin(bx+c)$$

$$= r\,e^{ax}\left(\cos(bx+c)\cos\theta - \sin(bx+c)\sin\theta\right)$$

$$= r\,e^{ax}\cos(bx+c+\theta)$$

Similarly, differentiating y w. r. t. x twice, thrice, ………, we get

$$y_2 = r^2\,e^{ax}\cos(bx+c+2\theta)$$

$$y_3 = r^3\,e^{ax}\cos(bx+c+3\theta)$$

………………………………

………………………………

Now we suppose that

$$y_n = r^n\,e^{ax}\cos(bx+c+n\theta) \qquad \text{.......(10)}$$

Differentiating both sides of equation (10) w. r. t. x, we get

$$y_{n+1} = r^n\left[ae^{ax}\cos(bx+c+n\theta) - be^{ax}\sin(bx+c+n\theta)\right]$$

$$= r^n e^{ax}\left[a\cos(bx+c+n\theta) - b\sin(bx+c+n\theta)\right]$$

$$= r^n e^{ax}\left[r\cos\theta\cos(bx+c+n\theta) - r\sin\theta\sin(bx+c+n\theta)\right]$$

(using $a = r\cos\theta,\ b = \sin\theta$)

$$= r^{n+1}e^{ax}\left[\cos(bx+c+n\theta)\cos\theta - \sin(bx+c+n\theta)\sin\theta\right]$$

$$= r^{n+1}e^{ax}\cos(bx+c+n\theta+\theta)$$

$$= r^{n+1}e^{ax}\cos\left(bx+c+(n+1)\theta\right)$$

Clearly, y_{n+1} is of the same form as y_n, given by equation (10). This $\Rightarrow$ If equation (10) is true for a particular value of n, then equation (10) is true for next higher value of n also.

But we have already proved that equation (10) is true for n = 1, 2, 3.

Since equation (10) is true for n = 3, it is true for n = 4 also. Similarly, since equation (10) is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, equation (10)) is true for every positive integer n,

i.e., $\boxed{y_n = r^n\,e^{ax}\cos(bx+c+n\,\theta),\ \text{where } r = \sqrt{a^2+b^2}\ \text{ and }\ \theta = \tan^{-1}\dfrac{b}{a}}$

Example 5 : Find n[th] differential coefficient of $\frac{1}{(1+x)^2}$.

Solution : Let $y = \frac{1}{(1+x)^2}$

Differentiating both sides w. r. t. x successively, we get

$$y_1 = (-2)(1+x)^{-3}$$

$$y_2 = (-2)(-3)\,(1+x)^{-4}$$

$$y_3 = (-2)(-3)(-4)(1+x)^{-5}$$

…………………………….

…………………………….

Proceeding in this way, we get

$$y_n = (-2)(-3)(-4)\ldots\ldots\ldots\ldots(-(n+1))(1+x)^{-(n+2)}$$

i.e., $$y_n = \frac{(-1)^n\,(n+1)!}{(1+x)^{(n+2)}}.$$

Example 6 : Find y_n if $y = \sin^2 x \cos^2 x$.

Solution : We have

$$y = \sin^2 x \cos^2 x = \frac{1}{4}(2\sin x \cos x)^2 = \frac{1}{4}(\sin 2x)^2 = \frac{1}{4}\sin^2 2x = \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)$$

i.e., $$y = \frac{1}{8} - \frac{1}{8}\cos 4x$$

This $\Rightarrow y_n = 0 - \frac{1}{8}4^n \cos\left(4x + \frac{n\pi}{2}\right)$

(using the formula : $y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$ if $y = \cos(ax+b)$)

i.e., $$y_n == -2^{n-2} \cos\left(4x + \frac{n\pi}{2}\right).$$

Example 7 : Find y_n if $y = e^x \sin^2 x$.

Solution : We have

$$y = e^x \sin^2 x = e^x \left(\frac{1-\cos 2x}{2}\right)$$

i.e., $$y = \frac{1}{2}e^x - \frac{1}{2}e^x \cos 2x$$

This $\Rightarrow y_n = \frac{1}{2}e^x - \frac{1}{2}(1^2+2^2)^{\frac{n}{2}} e^x \cos\left(2x + n\tan^{-1} 2\right)$

(using the formulae : $y_n = m^n e^{mx}$ if $y = e^{mx}$

and $y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$ if $y = e^{ax}\cos(bx+c)$)

i.e., $$y_n = \frac{1}{2}e^x\left[1 - (5)^{\frac{n}{2}} \cos\left(2x + n\tan^{-1} 2\right)\right].$$

Example 8 : If $y=\sin mx+\cos mx$, prove that $y_n = m^n\left[1+(-1)^n \sin 2mx\right]^{\frac{1}{2}}$.

Solution : We have

$$y=\sin mx+\cos mx$$

This $\Rightarrow$ $y_n = m^n \sin\left(mx+\frac{n\pi}{2}\right)+m^n\cos\left(mx+\frac{n\pi}{2}\right)$

$$\left(\begin{array}{ll}\text{using the formulae :} & y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right) \text{ if } y=\sin(ax+b) \\ \text{and} & y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right) \text{ if } y=\cos(ax+b)\end{array}\right)$$

i.e., $y_n = m^n\left\{\sin\left(mx+\frac{n\pi}{2}\right)+\cos\left(mx+\frac{n\pi}{2}\right)\right\}$.

This $\Rightarrow y_n = m^n\left[\left\{\sin\left(mx+\frac{n\pi}{2}\right)+\cos\left(mx+\frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$

$$= m^n\left[\sin^2\left(mx+\frac{n\pi}{2}\right)+\cos^2\left(mx+\frac{n\pi}{2}\right)+2\sin\left(mx+\frac{n\pi}{2}\right)\cos\left(mx+\frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$

$$= m^n\left[1+2\sin\left(mx+\frac{n\pi}{2}\right)\cos\left(mx+\frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$

$$= m^n\left[1+\sin 2\left(mx+\frac{n\pi}{2}\right)\right]^{\frac{1}{2}} \qquad \text{(using the formulae } 2\sin\theta\cos\theta=\sin 2\theta\text{)}$$

$$= m^n\left[1+\sin(2mx+n\pi)\right]^{\frac{1}{2}}$$

$$= m^n\left[1+\sin 2mx\cos n\pi+\cos 2mx\sin n\pi\right]^{\frac{1}{2}}$$

$$= m^n\left[1+(-1)^n\sin 2mx\right]^{\frac{1}{2}} \qquad \text{(since } \cos n\pi=(-1)^n \text{ and } \sin n\pi=0\text{)}$$

3. Method of finding nth derivative of an Algebraic Rational Function

In order to find the n^{th} derivative of an algebraic rational function, it is first required to check whether the denominator of the given algebraic rational function is resolvable into real linear factors or not.

If the denominator of the given algebraic rational function is resolvable into real linear factors, then the given algebraic rational function is resolved into partial fractions. In this case, the denominator of each partial fraction thus obtained consists of real linear factors, repeated or non-repeated, therefore the n^{th} derivative of the given algebraic rational function can be found out by directly using the formulae derived earlier for n^{th} derivative of some standard functions.

If the denominator of the given algebraic rational function is not resolvable into real linear factors, then its n^{th} derivative is found out by the application of De Moivre's theorem.

Example 9 : Find y_n if $y=\dfrac{x^4}{(x-1)(x-2)}$.

Solution : We have

$$y=\frac{x^4}{(x-1)(x-2)}=\frac{x^4}{x^2-3x+2}=x^2+3x+7+\frac{15x-14}{x^2-3x+2}$$

i.e., $y=x^2+3x+7+\dfrac{15x-14}{(x-1)(x-2)}$ ....(1)

Now let us resolve $\dfrac{15x-14}{(x-1)(x-2)}$ into partial fractions.

Let $\dfrac{15x-14}{(x-1)(x-2)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}$ ....(2)

This$\Rightarrow \dfrac{15x-14}{(x-1)(x-2)}=\dfrac{A(x-2)+B(x-1)}{(x-1)(x-2)}$

$\Rightarrow 15x-14 = A(x-2)+B(x-1)$ ....(3)

Putting $x=1$ on both sides both sides of equation (3), we get $A=-1$.

Putting $x=2$ on both sides both sides of equation (3), we get $B=16$.

Substituting the values of A and B in equation (2) and using equation (1), we get

$$y=x^2+3x+7+\frac{16}{x-2}-\frac{1}{x-1}$$

This $\Rightarrow y_n=0+0+0+16\left[\dfrac{(-1)^n n!}{(x-2)^{n+1}}\right]-\left[\dfrac{(-1)^n n!}{(x-1)^{n+1}}\right]$

$$\left(\text{usig the formula : } y_n=\frac{(-1)^n a^n n!}{(ax+b)^{n+1}} \text{ if } y=\frac{1}{ax+b}\right)$$

i.e., $y_n=(-1)^n n!\left[\dfrac{16}{(x-2)^{n+1}}-\dfrac{1}{(x-1)^{n+1}}\right]$.

Example 10 : Find y_n if $y=\dfrac{x^2}{(x-1)^2(x+2)}$.

Solution : We have

$$y=\frac{x^2}{(x-1)^2(x+2)} \qquad ....(1)$$

Let us resolve $\dfrac{x^2}{(x-1)^2(x+2)}$ into partial fractions.

Let $\dfrac{x^2}{(x-1)^2(x+2)}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+2}$ ....(2)

This $\Rightarrow \dfrac{x^2}{(x-1)^2(x+2)} = \dfrac{A(x-1)(x+2)+B(x+2)+C(x-1)^2}{(x-1)^2\,(x+2)}$

$$\Rightarrow x^2 = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \qquad ....(3)$$

Putting $x=1$ on both sides of equation (3), we get $B=1/3$.

Putting $x=-2$ on both sides of equation (3), we get $C=4/9$.

Equating the coefficients of x^2 from both sides of equation (3), we get

$$1 = A + C, \text{ which } \Rightarrow A = 1 - C = 1 - \frac{4}{9} = \frac{5}{9}.$$

Substituting the values of A, B and C in equation (2) and using equation (1), we get

$$y = \frac{5}{9}\left(\frac{1}{x-1}\right) + \frac{1}{3(x-1)^2} + \frac{4}{9}\left(\frac{1}{x+2}\right)$$

This $\Rightarrow y_n = \dfrac{5}{9}\left[\dfrac{(-1)^n n!}{(x-1)^{n+1}}\right] + \dfrac{1}{3}\left[\dfrac{(-1)^n (n+1)!}{(x-1)^{n+2}}\right] + \dfrac{4}{9}\left[\dfrac{(-1)^n n!}{(x+2)^{n+1}}\right].$

$$\left(\begin{array}{ll} \text{using the formulae:} & y_n = \dfrac{(-1)^n a^n n!}{(ax+b)^{n+1}} \text{ if } y = \dfrac{1}{ax+b} \\ \qquad\qquad\text{and} & y_n = \dfrac{(-1)^n a^n (n+1)!}{(ax+b)^{n+2}} \text{ if } y = \dfrac{1}{(ax+b)^2} \end{array}\right)$$

Example 11 : Find y_n if $y = \dfrac{1}{x^2+a^2}$.

Solution : We have

$$y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left[\frac{(x+ia)-(x-ia)}{(x+ia)(x-ia)}\right]$$

i.e., $\quad y = \dfrac{1}{2ia}\left[\dfrac{1}{(x-ia)} - \dfrac{1}{(x+ia)}\right]$

This $\Rightarrow y_n = \dfrac{1}{2ia}\left[\dfrac{(-1)^n n!}{(x-ia)^{n+1}} - \dfrac{(-1)^n n!}{(x+ia)^{n+1}}\right]$ $\left(\text{usig the formula : } y_n = \dfrac{(-1)^n a^n n!}{(ax+b)^{n+1}} \text{ if } y = \dfrac{1}{ax+b}\right)$

i.e., $\quad y_n = \dfrac{(-1)^n n!}{2ia}\left[\dfrac{1}{(x-ia)^{n+1}} - \dfrac{1}{(x+ia)^{n+1}}\right].$

Now put $x = r\cos\theta$ and $a = r\sin\theta$ so that $r = \sqrt{x^2+a^2}$ and $\theta = \tan^{-1}\dfrac{a}{x}$

Then we have

$$y_n = \frac{(-1)^n n!}{2ia}\left[\frac{1}{(r\cos\theta - i\,r\sin\theta)^{n+1}} - \frac{1}{(r\cos\theta + i\,r\sin\theta)^{n+1}}\right]$$

This $\Rightarrow y_n = \dfrac{(-1)^n n!}{2ia\, r^{n+1}}\left[\dfrac{1}{(\cos\theta - i\sin\theta)^{n+1}} - \dfrac{1}{(\cos\theta + i\sin\theta)^{n+1}}\right]$

$$= \frac{(-1)^n n!}{2ia\, r^{n+1}}\left[\frac{1}{(e^{-i\theta})^{n+1}} - \frac{1}{(e^{i\theta})^{n+1}}\right] \quad \left(\text{usig the formula: } e^{i\theta} = \cos\theta + i\sin\theta\right)$$

$$= \frac{(-1)^n n!}{2ia\, r^{n+1}}\left[\frac{1}{e^{-i(n+1)\theta}} - \frac{1}{e^{i(n+1)\theta}}\right]$$

$$= \frac{(-1)^n n!}{2ia\, r^{n+1}}\left[e^{i(n+1)\theta} - e^{-i(n+1)\theta}\right]$$

$$= \frac{(-1)^n n!}{2ia\, r^{n+1}}\left[\{\cos(n+1)\theta + i\sin(n+1)\theta\} - \{\cos(n+1)\theta - i\sin(n+1)\theta\}\right]$$

$$= \frac{(-1)^n n!}{2ia\, r^{n+1}}\left[2i\sin(n+1)\theta\right]$$

$$= \frac{(-1)^n n!}{a\left(\dfrac{a}{\sin\theta}\right)^{n+1}}\sin(n+1)\theta \qquad \left(\text{since } a = r\sin\theta, \text{ i.e., } r = \frac{a}{\sin\theta}\right)$$

i.e., $\quad y_n = \dfrac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta.$

Example 12 : Find y_n if $y = \tan^{-1} x.$

Solution : We have

$$y = \tan^{-1} x$$

Differentiating both sides w. r. t. x , we get

$$y_1 = \frac{1}{1+x^2}$$

This $\Rightarrow y_1 == \dfrac{1}{(x+i)(x-i)} = \dfrac{1}{2i}\left[\dfrac{(x+i)-(x-i)}{(x+i)(x-i)}\right]$

i.e., $\quad y_1 = \dfrac{1}{2i}\left[\dfrac{1}{(x-i)} - \dfrac{1}{(x+i)}\right]$

Differentiating y_1 w. r. t. x $(n-1)$ times, we get

$$y_n = \frac{1}{2i}\left[\frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{(x-i)}\right) - \frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{(x+i)}\right)\right]$$

This $\Rightarrow y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$

$\left(\text{using the formula : } y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}} \text{ if } y = \frac{1}{ax+b}\right)$

i.e., $y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$.

Now put $x = r\cos\theta$ and $1 = r\sin\theta$ so that $r = \sqrt{x^2+1}$ and $\theta = \tan^{-1}\frac{1}{x}$

Then we have

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(r\cos\theta - i\,r\sin\theta)^n} - \frac{1}{(r\cos\theta + i\,r\sin\theta)^n}\right]$$

This $\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{2\,i\,r^n}\left[\frac{1}{(\cos\theta - i\sin\theta)^n} - \frac{1}{(\cos\theta + i\sin\theta)^n}\right]$

$$= \frac{(-1)^{n-1}(n-1)!}{2\,i\,r^n}\left[\frac{1}{(e^{-i\theta})^n} - \frac{1}{(e^{i\theta})^n}\right] \quad (\text{using } e^{i\theta} = \cos\theta + i\sin\theta)$$

$$= \frac{(-1)^{n-1}(n-1)!}{2\,i\,r^n}\left[\frac{1}{e^{-in\theta}} - \frac{1}{e^{in\theta}}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2\,i\,r^n}\left[e^{in\theta} - e^{-in\theta}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2\,i\,r^n}\left[\{\cos n\theta + i\sin n\theta\} - \{\cos n\theta - i\sin n\theta\}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2\,i\,r^n}\left[2i\sin n\theta\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{\left(\frac{1}{\sin\theta}\right)^n}\sin n\theta \quad \left(\text{since } 1 = r\sin\theta, \text{ i.e., } r = \frac{1}{\sin\theta}\right)$$

i.e., $y_n = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$.

4. Leibnitz's Theorem

Leibnitz's Theorem is used to find the n^{th} derivative of the product of two functions. Its statement and proof are as follows :

Statement : If u and v are two functions of x, possessing derivatives of the n^{th} order, then

$$(uv)_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \ldots\ldots\ldots + {}^nC_r\, u_{n-r}v_r + \ldots\ldots\ldots + {}^nC_n\, u\, v_n.$$

Proof : Let $y = uv$, where u and v are functions of x.

Differentiating both sides w. r. t. x, we get

$$y_1 = u_1 v + u v_1$$

Clearly, y_1 can be written as

$$y_1 = {}^1C_0 u_1 v + {}^1C_1 u v_1 \qquad \left(\because {}^1C_0 = 1,\ {}^1C_1 = 1\right)$$

This $\Rightarrow$ Leibnitz's Theorem is true for $n = 1$.

Differentiating y_1 w. r. t. x, we get

$$y_2 = \left(u_2 v + u_1 v_1\right) + \left(u_1 v_1 + u v_2\right) = u_2 v + 2 u_1 v_1 + u v_2$$

Clearly, y_2 can be written as

$$y_2 = {}^2C_0 u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2 \qquad \left(\because {}^2C_0 = 1,\ {}^2C_1 = 2,\ {}^2C_2 = 1\right)$$

This $\Rightarrow$ Leibnitz's Theorem is true for $n = 2$.

Differentiating y_2 w. r. t. x, we get

$$y_3 = \left(u_3 v + u_2 v_1\right) + 2\left(u_2 v_1 + u_1 v_2\right) + \left(u_1 v_2 + u v_3\right) = u_3 v + 3 u_2 v_1 + 3 u_1 v_2 + u v_3$$

Clearly, y_3 can be written as

$$y_3 = {}^3C_0 u_3 v + {}^3C_1 u_2 v_1 + {}^3C_2 u_1 v_2 + {}^3C_3 u v_3 \qquad \left(\because {}^3C_0 = 1,\ {}^3C_1 = 3,\ {}^3C_2 = 3,\ {}^3C_3 = 1\right)$$

This $\Rightarrow$ Leibnitz's Theorem is true for $n = 3$.

Now we suppose that Leibnitz's Theorem is true for a particular value of n, say, m, i.e., we suppose that

$$y_m = {}^mC_0 u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \ldots\ldots\ldots + {}^mC_{r-1} u_{m-r+1} v_{r-1} + {}^mC_r u_{m-r} v_r$$

$$+ \ldots\ldots\ldots + {}^mC_{m-1} u_1 v_{m-1} + {}^mC_m u v_m \qquad \ldots..(1)$$

Differentiating both sides of equation (1) w. r. t. x, we get

$$y_{m+1} = {}^mC_0 \left(u_{m+1} v + u_m v_1\right) + {}^mC_1 \left(u_m v_1 + u_{m-1} v_2\right) + {}^mC_2 \left(u_{m-1} v_2 + u_{m-2} v_3\right)$$

$$+ \ldots\ldots\ldots + {}^mC_{r-1} \left(u_{m-r+2} v_{r-1} + u_{m-r+1} v_r\right) + {}^mC_r \left(u_{m-r+1} v_r + u_{m-r} v_{r+1}\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots + {}^mC_{m-1} \left(u_2 v_{m-1} + u_1 v_m\right) + {}^mC_m \left(u_1 v_m + u v_{m+1}\right)$$

This $\Rightarrow$ $y_{m+1} = {}^mC_0 u_{m+1} v + \left({}^mC_0 + {}^mC_1\right) u_m v_1 + \left({}^mC_1 + {}^mC_2\right) u_{m-1} v_2 + \ldots\ldots\ldots\ldots + $

$$\left({}^mC_{r-1} + {}^mC_r\right) u_{m-r+1} v_r + \ldots\ldots + \left({}^mC_{m-1} + {}^mC_m\right) u_1 v_m + {}^mC_m u v_{m+1} \qquad \ldots..(2)$$

But we know that $\quad {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

Putting $r = 1, 2, 3, \ldots\ldots, m$ successively in this well-known result, we get

$${}^mC_0 + {}^mC_1 = {}^{m+1}C_1,\ {}^mC_1 + {}^mC_2 = {}^{m+1}C_2, \ldots\ldots\ldots, {}^mC_{m-1} + {}^mC_m = {}^{m+1}C_m$$

Also, we know that $\quad {}^mC_0 = {}^{m+1}C_0 = 1$ and $\quad {}^mC_m = {}^{m+1}C_{m+1} = 1.$

Substituting these values in equation (2), we get

$$y_{m+1} = {}^{m+1}C_0\, u_{m+1}v + {}^{m+1}C_1\, u_m v_1 + {}^{m+1}C_2\, u_{m-1}v_2 + \ldots\ldots\ldots\ldots\ldots$$

$$+ {}^{m+1}C_r\, u_{m+1-r}v_r + \ldots\ldots + {}^{m+1}C_m\, u_1 v_m + {}^{m+1}C_{m+1}\, uv_{m+1} \qquad \ldots..(3)$$

Equation (3) $\Rightarrow$ Leibnitz's theorem is true for $n = m+1$.

Thus we have proved that if Leibnitz's Theorem is true for a particular value of n, say, m, then it is true for next higher value of n, i.e., for m+1 also.

But we have already proved that Leibnitz's Theorem is true for n = 1, 2, 3.

Since Leibnitz's Theorem is true for n = 3, it is true for n = 4 also. Similarly, since Leibnitz's Theorem is true for n = 4, it is true for n = 5 also and so on.

Therefore, by mathematical induction, Leibnitz's Theorem is true for every positive integer n, i.e., we have

$$\boxed{(uv)_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \ldots\ldots\ldots + {}^nC_r\, u_{n-r}v_r + \ldots\ldots\ldots + {}^nC_n\, u\, v_n}$$

Example 13 : Find y_n if $y = x^2 e^{ax}$.

Solution : By Leibnitz's Theorem, if $y = uv$, then

$$y_n = u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \ldots\ldots\ldots\ldots + {}^nC_n\, u\, v_n \qquad \ldots(1)$$

We are given that $y = x^2 e^{ax}$.

Therefore, we can take $u = e^{ax}$, $v = x^2$.

Then we have $u_1 = ae^{ax}$, $u_2 = a^2 e^{ax}$, $u_3 = a^3 e^{ax}$, ..., $u_n = a^n e^{ax}$

and $\quad v_1 = 2x,\ v_2 = 2,\ v_3 = 0,\ v_2 = 0, \ldots\ldots\ldots, v_n = 0.$

Substituting these values in equation (1), we get

$$y_n = a^n e^{ax} x^2 + {}^nC_1\, a^{n-1} e^{ax}(2x) + {}^nC_2\, a^{n-2} e^{ax}(2) + 0 + 0 + \ldots..$$

$$= a^n e^{ax} x^2 + n\, a^{n-1} e^{ax}(2x) + \frac{n(n-1)}{2} a^{n-2} e^{ax}(2)$$

$$= a^{n-2} e^{ax}[a^2x^2 + 2n\,ax + n(n-1)].$$

Example 14 : Find y_n if $y = x^2 \log x$, where $n \geq 3$.

Solution : By Leibnitz's Theorem, if $y = uv$, then

$$y_n = u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \ldots\ldots\ldots\ldots + {}^nC_n\, u\, v_n \qquad \ldots(1)$$

We are given that $y = x^2 \log x$.

Therefore, we can take $u = \log x$, $v = x^2$.

Then we have

$$u_1 = \frac{1}{x},\ \ u_2 = -\frac{1}{x^2}, u_3 = \frac{(-1)(-2)}{x^3}, \dots\dots\dots, u_n = \frac{(-1)(-2)\dots\dots\dots\big(-(n-1)\big)}{x^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

and $v_1 = 2x,\ v_2 = 2,\ v_3 = 0,\ v_2 = 0, \dots\dots\dots\dots\dots, v_n = 0.$

Substituting these values in equation (1), we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}x^2 + {}^nC_1\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}(2x) + {}^nC_2\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}(2) + 0 + 0 + \dots\dots$$

$$= \frac{(-1)^{n-3}}{x^{n-2}}[(-1)^2(n-1)! + {}^nC_1(-1)(n-2)!(2) + {}^nC_2(n-3)!(2)]$$

$$= \frac{(-1)^{n-3}}{x^{n-2}}[(n-1)! - n(n-2)!(2) + \frac{n(n-1)}{2}(n-3)!(2)]$$

$$= \frac{(-1)^{n-3}}{x^{n-2}}[(n-1)(n-2)(n-3)! - 2n(n-2)(n-3)! + n(n-1)(n-3)!]$$

$$= \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}[(n-1)(n-2) - 2n(n-2) + n(n-1)]$$

$$= \frac{(-1)^{n-1}(n-3)!}{(-1)^2x^{n-2}}[(n^2-3n+2) - (2n^2-2n) + (n^2-n)]$$

$$= \frac{(-1)^{n-1}(n-3)!}{x^{n-2}}(2)$$

$$= \frac{2(-1)^{n-1}(n-3)!}{x^{n-2}}$$

Example 15 : If $y = (\sin^{-1}x)^2$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$

Solution : Given that $y = (\sin^{-1}x)^2$.

Differentiating both sides w. r. t. x, we get

$$y_1 = 2\left(\sin^{-1}x\right)\left(\frac{1}{\sqrt{1-x^2}}\right)$$

This $\Rightarrow y_1\sqrt{1-x^2} = 2\sin^{-1}x$

$\Rightarrow y_1{}^2\left(1-x^2\right) = 4(\sin^{-1}x)^2$ (on squaring both sides)

$\Rightarrow y_1{}^2\left(1-x^2\right) = 4y$ $\left(\because y = (\sin^{-1}x)^2 \text{ given}\right)$

Differentiating both sides w. r. t. x, we get

$$2y_1y_2(1-x^2) + y_1{}^2(-2x) = 4y_1$$

This $\Rightarrow 2y_1[y_2(1-x^2) - xy_1] = 4y_1$

$\Rightarrow y_2(1-x^2) - xy_1 = 2$

Differentiating both sides n times w. r. t. x and using Leibnitz's Theorem, we get

$$[y_{n+2}(1-x^2) + {}^nC_1y_{n+1}(-2x) + {}^nC_2y_n(-2)] - [y_{n+1}x + {}^nC_1y_n(1)] = 0$$

(by Leibnitz's Theorem, $(uv)_n = u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \ldots\ldots\ldots + {}^nC_n\, u\, v_n$)

This $\Rightarrow \left[y_{n+2}(1-x^2) + n\, y_{n+1}(-2x) + \dfrac{n(n-1)}{2} y_n(-2) \right] - \left[y_{n+1}x + n\, y_n \right] = 0$

$\Rightarrow y_{n+2}(1-x^2) - (2n+1)xy_{n+1} - [n(n-1)+n]y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0.$

Example 16 : If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x,$ prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$

Solution: Given that $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x.$

Differentiating both sides w. r. t. $x,$ we get

$$\frac{1}{m} y^{\frac{1}{m}-1} y_1 + \left(-\frac{1}{m}\right) y^{-\frac{1}{m}-1} y_1 = 2$$

This $\Rightarrow \dfrac{1}{m} \dfrac{y^{\frac{1}{m}}}{y} y_1 - \dfrac{1}{m} \dfrac{y^{-\frac{1}{m}}}{y} y_1 = 2$

$\Rightarrow y_1 \left[y^{\frac{1}{m}} - y^{-\frac{1}{m}} \right] = 2my$

$\Rightarrow y_1^2 \left[y^{\frac{1}{m}} - y^{-\frac{1}{m}} \right]^2 = 4m^2 y^2$ (on squaring both sides)

$\Rightarrow y_1^2 \left[\left(y^{\frac{1}{m}} + y^{-\frac{1}{m}} \right)^2 - 4 y^{\frac{1}{m}} y^{-\frac{1}{m}} \right] = 4m^2 y^2$

$\Rightarrow y_1^2 \left[(2x)^2 - 4 \right] = 4m^2 y^2$ $\left(\because y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x \text{ given} \right)$

$\Rightarrow 4y_1^2 (x^2-1) = 4m^2 y^2$

$\Rightarrow y_1^2 (x^2-1) = m^2 y^2$

Differentiating both sides w. r. t. $x,$ we get

$$2y_1 y_2 (x^2-1) + y_1^2 (2x) = m^2 (2yy_1)$$

This $\Rightarrow 2y_1[y_2(x^2-1) + xy_1] = 2y_1(m^2 y)$

$\Rightarrow y_2(x^2-1) + xy_1 = m^2 y.$

Differentiating both sides n times w. r. t. x and using Leibnitz's Theorem, we get

$$[y_{n+2}(x^2-1) + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n(2)] + [y_{n+1}x + {}^nC_1 y_n(1)] = m^2 y_n$$

(by Leibnitz's Theorem, $(uv)_n = u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \ldots\ldots\ldots + {}^nC_n\, u\, v_n$)

This $\Rightarrow \left[y_{n+2}(x^2-1) + n\, y_{n+1}(2x) + \dfrac{n(n-1)}{2} y_n(2) \right] + \left[y_{n+1}x + n\, y_n \right] = m^2 y_n$

$$\Rightarrow y_{n+2}(x^2-1)+(2n+1)xy_{n+1}+[n\,(n-1)+n]y_n = m^2 y_n$$

$$\Rightarrow (x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n = 0.$$

Exercises

1. If $y = A\sin mx + B\cos mx$, prove that $y_2 = -m^2 y$.
2. If $y = \sin(\sin x)$, prove that $y_2 + y_1 \tan x + y\cos^2 x = 0$.
3. If $y = \log\left(\dfrac{x}{a+bx}\right)^x$, prove that $x^3 y_2 = (y - x\,y_1)^2$.
4. If $y = ax^2 + 2hxy + by^2 = 1$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{h^2-ab}{(hx+by)^3}$.
5. Find y_n if $y = \dfrac{1}{(x-1)^3(x-2)}$.
6. Find y_n if $y = \dfrac{x^n}{(1+x)}$.
7. Find y_n if $y = x^2 e^{2x}\sin x$.
8. If $y = e^{x^2}$, prove that $y_{n+1} - 2xy_n - 2n\,y_{n-1} = 0$.
9. If $y = \sin m(\sin^{-1}x)$, prove that $\lim\limits_{x\to 0}\dfrac{y_{n+2}}{y_n} = n^2 - m^2$.
10. If $y = e^{a\tan^{-1}x} = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots$, then prove that $(n+2)a_{n+2} + na_n = a_{n+1}$.

Answers

5. $y_n = \dfrac{(-1)^n\,n!}{(x-2)^{n+1}} - \dfrac{(-1)^n\,n!}{(x-1)^{n+1}} - \dfrac{(-1)^n\,(n+1)!}{(x-1)^{n+2}} - \dfrac{(-1)^n\,(n+2)!}{2\,(x-1)^{n+3}}$

6. $y_n = \dfrac{n!}{(x+1)^{n+1}}$

7. $y_n = 5^{\frac{n}{2}}\,x^2 e^{2x}\sin\left(x + n\tan^{-1}\dfrac{1}{2}\right) + 2\,n\,x\,5^{\frac{n-1}{2}}\,e^{2x}\sin\left(x + (n-1)\tan^{-1}\dfrac{1}{2}\right)$
$\quad + n\,(n-1)\,5^{\frac{n-2}{2}}\,e^{2x}\sin\left(x + (n-2)\tan^{-1}\dfrac{1}{2}\right)$