Topic: Special Theory of Relativity, Intervals Spacetime and Proper Time

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Intervals in Spacetime

Let us express mathematically the constancy of c in all frames. An *event* is specified by the time and place where it occurs. Thus, an event is specified by *four* coordinates,(t, x, y, z). The four-dimensional space spanned by these coordinates is called *spacetime*.
The *interval* between two events in spacetime at (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) is defined to be

$$s_{12} = \sqrt{c^2(t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2}\ . \qquad (1)$$

For two events separated by an infinitesimal amount, the interval ds is infinitesimal, with

$$ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2\ . \qquad (2)$$

when the two events denote the emission and reception of an electromagnetic signal, we have $ds^2 = 0$. This must be true in any frame, as the fact the invariance of c, hence since ds and ds' are differentials of the same order, we must have $ds'^2 = ds^2$. This last result requires homogeneity and isotropy of space as well. Finally, if infinitesimal intervals are invariant, then integrating we obtain $s = s'$, and we conclude that *the interval between two space-time events is the same in all inertial frames*.

When $s_{12}^2 > 0$, the interval is said to be *time-like*. For timelike intervals, we can always find a reference frame in which the two events occur at the same *locations*. As an example, let a passenger sitting on a train. Event #1 is the passenger yawning at time t_1. Event #2 is the passenger yawning again at some later time t_2. To an observer sitting in the train station, the two events take place at different locations, but in the frame of the passenger, they occur at the same location.

When ${s^2}_{12} < 0$, the interval is said to be *space-like*.

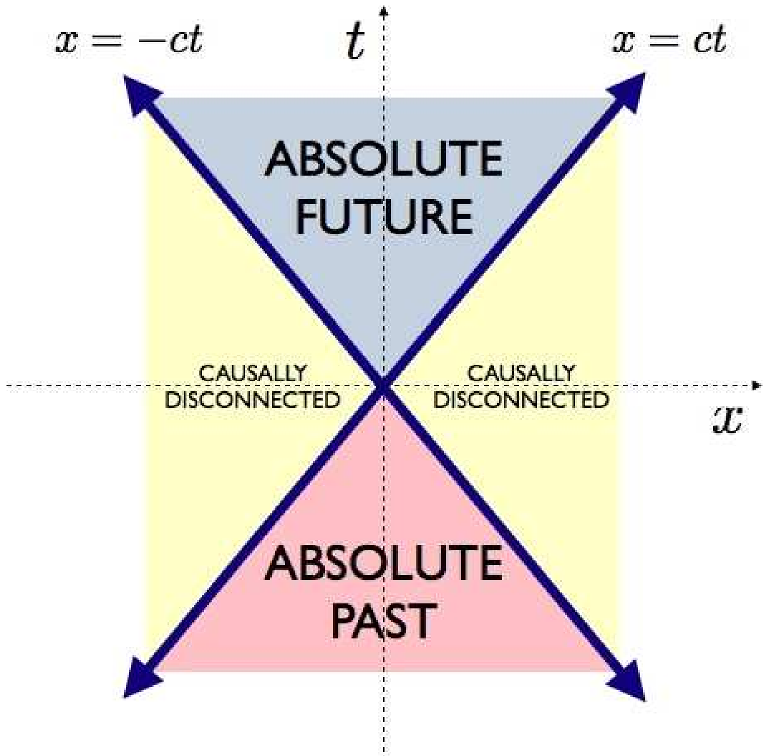

Figure : A (1 + 1)–dimensional light cone. The forward light cone consists of timelike events with $\Delta t > 0$. The backward light cone consists of timelike events with $\Delta t < 0$. The causally disconnected regions are time-like, and intervals connecting the origin to any point on the light cone itself are light-like.

moment, in the frame of the reader, the North and South poles of the earth are separated by a space-like interval. If the interval between two events is space-like, a reference frame can always be found in which the events are simultaneous.

An interval with $s_{12} = 0$ is said to be *light-like*.

This leads to the concept of the *light cone*, shown in figure. Let an event E. In the frame of an inertial observer, all events with $s^2 > 0$ and $\Delta t > 0$ are in E's *forward light cone* and are part of his *absolute future*. Events with $s^2 > 0$ and $\Delta t < 0$ lie in E's *backward light cone* are are part of his *absolute past*. Events with spacelike separations $s^2 < 0$ are *causally disconnected* from E. Two events which are causally disconnected can not possible influence each other. Uniform rectilinear motion is represented by a line $t = x/v$ with constant slope. If $v < c$, this line is contained within E's light cone. E is potentially influenced by all events in its backward light cone, *i.e.* its absolute past. It is impossible to find a frame of reference which will transform past into future, or spacelike into timelike intervals.

Proper time

Proper time is the time read on a clock traveling with a moving observer. Let two observers, one at rest and one in motion. If dt is the differential time elapsed in the rest frame, then

$$
\begin{aligned}
ds^2 &= c^2\,dt^2 - dx^2 - dy^2 - dz^2 \qquad (3)\\
&= c^2\,dt'^2 \;, \qquad (4)
\end{aligned}
$$

where dt' is the differential time elapsed on the moving clock. Thus,

$$dt' = dt\,\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} \;, \tag{5}$$

and the time elapsed on the moving observer's clock is

$$t'_2 - t'_1 = \int_{t_1}^{t_2} dt\,\sqrt{1 - \frac{\boldsymbol{v}^2(t)}{c^2}} \;. \tag{6}$$

Thus, *moving clocks run slower.* This is an essential feature which helps in understanding many important aspects of particle physics. A particle with a brief lifetime can, by moving at speeds close to c, appear to an observer in our frame to be long-lived. It is customary to define two dimensionless measures of a particle's velocity:

$$\boldsymbol{\beta} \equiv \frac{\boldsymbol{v}}{c} \qquad , \qquad \gamma \equiv \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}} \;. \tag{7}$$

As $v \to c$, we have $\beta \to 1$ and $\gamma \to \infty$.

Suppose we wish to compare the elapsed time on two clocks. We keep one clock at rest in an inertial frame, while the other executes a closed path in space, returning to its initial location after some interval of time. When the clocks are compared, the moving clock will show a smaller elapsed time. This is stated as the "twin paradox." The total elapsed time on a moving clock is given by

$$\tau = \frac{1}{c}\int_a^b ds \;, \tag{8}$$

where the integral is taken over the *world line* of the moving clock.