Reflection at a Conducting Surface

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In the case of conductors the free current J_f is not zero. In fact, according to Ohm's law, the (free) current density in a conductor is proportional to the electric field:

$$\mathbf{J} = \sigma \mathbf{E} \tag{1}$$

With this, Maxwell's equations for linear media assume the form

$$\begin{aligned}
\boldsymbol{\nabla}.\mathbf{E} &= \frac{\rho_f}{\epsilon} \\
\boldsymbol{\nabla}.\mathbf{B} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\boldsymbol{\nabla} \times \mathbf{B} &= \mu\sigma\mathbf{E} + \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

Now the continuity equation for free charge,

$$\boldsymbol{\nabla}.\mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}, \tag{2}$$

together with Ohm's law and Gauss's law (First Maxwell's equation), gives

$$\frac{\partial \rho_f}{\partial t} = -\sigma(\boldsymbol{\nabla}.\mathbf{E}) = -\frac{\sigma}{\epsilon}\rho_f \tag{3}$$

for a homogeneous linear medium, from which it follows that

$$\rho_f(t) = \rho_f(0)e^{-\sigma/\epsilon t}. \tag{4}$$

Thus any initial free charge density $\rho_f(0)$ dissipates in a characteristic time $\tau \equiv \epsilon/\sigma$. This reflects the familiar fact that if you put some free charge on a conductor, it will flow out to the edges. The time constant τ affords a measure of how "good" a conductor is: For a "perfect" conductor $\sigma = \infty$ and $\tau = 0$; for a "good" conductor, τ is much less than the other relevant times in the problem (in oscillatory systems, that means $\tau \ll 1/\omega$); for a "poor" conductor, τ is *greater* than the characteristic times in the problem ($\tau \gg 1/\omega$).

Hence, the boundary conditions which were used to analyze reflection and refraction at an interface between two dielectrics do not hold in the presence of free charges and currents. Instead, we have the more general relations

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f \tag{5}$$

$$B_1^{\perp} - B_2^{\perp} = 0 \tag{6}$$

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \tag{7}$$

$$\frac{\mathbf{B}_1^{\parallel}}{\mu_1} - \frac{\mathbf{B}_2^{\parallel}}{\mu_2} = \mathbf{K}_f \times \hat{\mathbf{n}} \tag{8}$$

where σ_f (not to be confused with conductivity) is the free surface charge, $\mathbf{K}_f$ the free surface current, and $\hat{\mathbf{n}}$ (not to be confused with the polarization of the wave) is a unit vector perpendicular to the surface, pointing from medium (2) into medium (1). For ohmic conductors ($\mathbf{J_f} = \sigma\mathbf{E}$) there can be no free surface current, since this would require an infinite electric field at the boundary.

Suppose now that the xy plane forms the boundary between a nonconducting linear medium (1) and a conductor (2). A monochromatic plane wave, of frequency ω traveling in the Z direction and polarized in the x direction, approaches from the left, as in Fig. 1

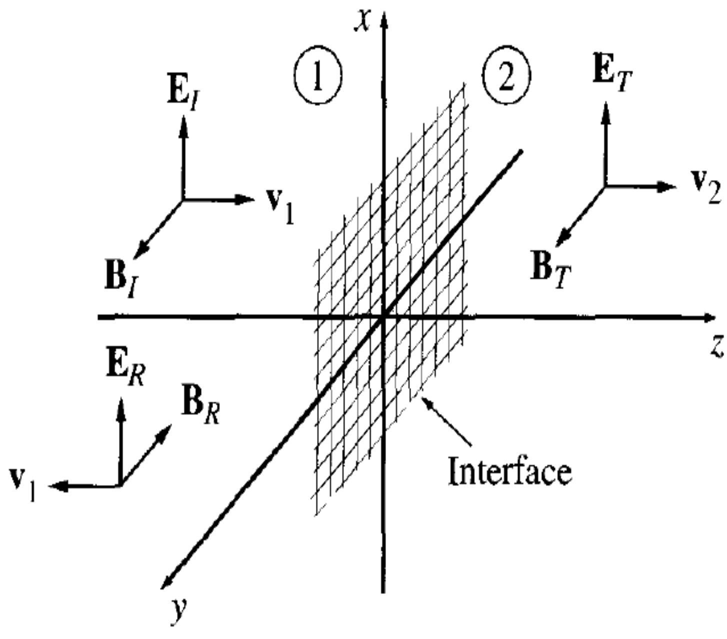

Figure 1:

$$\mathbf{E}_I(z,t) = E_{0_I} e^{(k_1 z - \omega t)} \hat{\mathbf{x}} \tag{9}$$

$$\mathbf{B}_I(z,t) = B_{0_I} e^{(k_1 z - \omega t)} \hat{\mathbf{y}} = \frac{E_{0_I}}{v_1} e^{(k_1 z - \omega t)} \hat{\mathbf{y}} \tag{10}$$

This incident wave gives rise to a reflected wave,

$$\mathbf{E}_R(z,t) = E_{0_R} e^{(-k_1 z - \omega t)} \hat{\mathbf{x}} \tag{11}$$

$$\mathbf{B}_R(z,t) = -\frac{E_{0_R}}{v_1} e^{(-k_1 z - \omega t)} \hat{\mathbf{y}} \tag{12}$$

which travels back to the left in medium (1), and a transmitted wave

$$\mathbf{E}_T(z,t) = E_{0_T} e^{(k_2 z - \omega t)} \hat{\mathbf{x}} \tag{13}$$

$$\mathbf{B}_T(z,t) = \frac{E_{0_T}}{v_2} e^{(k_2 z - \omega t)} \hat{\mathbf{y}} \tag{14}$$

Or,

$$\mathbf{B}_T(z,t) = \frac{k_2}{\omega} E_{0_T} e^{(k_2 z - \omega t)} \hat{\mathbf{y}} \because v_2 = \omega / k_2 \tag{15}$$

which continues on the the right in medium (2) and is attenuated as it penetrates into the conductor.

At $z = 0$, the combined wave in medium (1) must join the wave in medium (2), pursuant to the boundary conditions in equations (5)-(8). Since $E\perp = 0$ on both sides, boundary condition (5) yields $\sigma_f = 0$. Since $B\perp = 0$, boundary condition (6) is automatically satisfied. Meanwhile, boundary condition (7) gives

$$E_{0_I} + E_{0_R} = E_{0_T} \tag{16}$$

and boundary condition (8) (with $\mathbf{K}_f = 0$) says

$$\frac{1}{\mu_1 v_1}(E_{0_I} - E_{0_R}) - \frac{k_2}{\mu_2 \omega} E_{0_T} = 0 \tag{17}$$

or

$$E_{0_I} - E_{0_R} = \beta E_{0_T} \tag{18}$$

where

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 \omega} k_2. \tag{19}$$

It follows that

$$E_{0_R} = \frac{1-\beta}{1+\beta} E_{0_I} \tag{20}$$

$$E_{0_T} = \frac{2}{1+\beta} E_{0_I} \tag{21}$$

What fraction of the incident energy is reflected, and what fraction is transmitted? The intensity (average power per unit area) is

$$I = \frac{1}{2} \epsilon v E_0^2 \tag{22}$$

Hence the reflection and transmission coefficients are

$$R = \frac{I_R}{I_I} = \frac{\epsilon_1 v_1 E_{0_R}^2}{\epsilon_1 v_1 E_{0_I}^2} = \left(\mid \frac{1-\beta}{1+\beta} \mid \right)^2 \tag{23}$$

$$\tag{24}$$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2 E_{0_T}^2}{\epsilon_1 v_1 E_{0_I}^2} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \left(\mid \frac{2}{1+\beta} \mid \right)^2 \tag{25}$$

These results are formally identical to the ones that apply at the boundary between *nonconductors* but β is now a complex number.

For a *perfect* conductor $(\sigma = \infty), k_2 = \infty$, so β is infinite, and

$$E_{0_R} = -E_{0_I} \tag{26}$$

$$E_{0_T} = 0. \tag{27}$$

In this case the wave is totally reflected because E_{0_T} is zero in equation (27) , with a 180°phase shift because of minus sign in the equation (26). (That's why excellent conductors make good mirrors.

In practice, you paint a thin coating of silver onto the back of a pane of glass-the glass has nothing to do with the *reflection*; it's just there to support the silver and to keep it from tarnishing. Since the skin depth in silver at optical frequencies is on the order of $100\mathring{A}$, you don't need a very thick layer.)

Example: Calculate the reflection coefficient for light at an air-to-silver interface ($\mu_1 = \mu_2 = \mu_0, \epsilon_1 = \epsilon_0, \sigma = 6 \times 10^7 (\Omega \text{m})^{-1}$, at optical frequencies $\omega = 4 \times 10^{15} s^{-1}$.

Solution: Since the reflection coefficient is:

$$R = \left(\mid \frac{1-\beta}{1+\beta} \mid \right)^2 \tag{28}$$

where

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 \omega} k_2. \tag{29}$$

We know from the propagation of electromagnetic waves in a conducting material that the propagation vector in a conducting medium is

$$\tilde{k} = k + i\kappa,$$

where

$$k \equiv \omega \sqrt{\frac{\epsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1 \right]^{1/2}, \quad \kappa \equiv \omega \sqrt{\frac{\epsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right]^{1/2}.$$

Since k_2, the propagation vector is a complex number in a conductor, let

$$k_2 = a + ib. \tag{30}$$

Since silver is a good conductor, $\sigma \gg \epsilon\omega$,

$$a \cong b \cong \sqrt{\frac{\sigma\omega\mu_2}{2}}. \tag{31}$$

Hence

$$\beta = \frac{\mu_1 v_1}{\mu_2 \omega} \sqrt{\frac{\sigma\omega\mu_2}{2}} (1+i) = \mu_1 v_1 \sqrt{\frac{\sigma}{2\mu_2\omega}} (1+i) \tag{32}$$

Let

$$\gamma = \mu_1 v_1 \sqrt{\frac{\sigma}{2\mu_2\omega}} \tag{33}$$

Since, medium 1 is air, $v_1 = c$ and it is given that $mu_1 = \mu_2 = \mu_0$,

$$\gamma = \mu_0 c \sqrt{\frac{\sigma}{2\mu_0\omega}} = c\sqrt{\frac{\sigma\mu_0}{2\omega}} \tag{34}$$

Given: $\sigma = 6 \times 10^7 (\Omega \mathrm{m})^{-1}, \omega = 4 \times 10^{15} s^{-1}$ and we know that

$\mu_0 = 4\pi \times 10^{-7}, c = 3 \times 10^8,$

$$\Rightarrow \gamma = 3 \times 10^8 \sqrt{\frac{6 \times 10^7 \times 4\pi \times 10^{-7}}{2 \times 4 \times 10^{15}}} = 29. \tag{35}$$

Now

$$\because \beta = \gamma(1+i) \tag{36}$$

$$\Rightarrow R = \left| \left(\frac{1-\beta}{1+\beta}\right) \right|^2 = \left(\frac{1-\beta}{1+\beta}\right)\left(\frac{1-\beta^*}{1+\beta^*}\right) \tag{37}$$

Or,

$$\Rightarrow R = \left(\frac{1-\gamma(1+i)}{1+\gamma(1+i)}\right)\left(\frac{1-\gamma(1-i)}{1+\gamma(1-i)}\right) \tag{38}$$

Or,

$$R = \frac{(1-\gamma)^2 + \gamma^2)}{(1+\gamma)^2 + \gamma^2)} \tag{39}$$

From equation (35),

$$\gamma = 29, \tag{40}$$

$$\Rightarrow R = 0.93 \tag{41}$$

Evidently 93% light is reflected.