Partial Differentiation

(For B.Sc./B.A. Part-I, Hons. And Subsidiary Courses of Mathematics)

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Contents

1. Definitions and Notations

Let $u = f(x, y)$ be a function of two independent variables x and y.

Then the partial derivative of u w. r. t. x is denoted by $\frac{\partial u}{\partial x}$ or u_x and this is defined as the first derivative of u w. r. t. x only, treating the other variable y as constant.

Therefore if $u = f(x, y)$, then, by definition,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{f(x+h,\ y) - f(x,\ y)}{h}$$

provided the above limit exists.

Similarly, the partial derivative of u w. r. t. y is denoted by $\frac{\partial u}{\partial y}$ or u_y and this is defined as the first derivative of u w. r. t. y only, treating the other variable x as constant.

Therefore if $u = f(x, y)$, then, by definition,

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{f(x,\ y+k) - f(x, y)}{k}$$

provided the above limit exists.

In a similar manner, if $u = f(x, y, z)$ be a function of three independent variables $x, y, z,$ then we can define $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$. In this case, $\frac{\partial u}{\partial x}$ means the first derivative of u w. r. t. x only, treating the remaining variables y, z as constants; $\frac{\partial u}{\partial y}$ means the first derivative of u w. r. t. y only, treating the remaining variables z, x as constants and $\frac{\partial u}{\partial z}$ means the first derivative of u w. r. t. z only, treating the remaining variables x, y as constants, i.e.,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{f(x+h,\ y,\ z) - f(x,\ y,\ z)}{h}$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{f(x,\ y+k,\ z) - f(x,\ y,\ z)}{k}$$

and $$\frac{\partial u}{\partial z} = \lim_{l\to 0} \frac{f(x,\ y,\ z+l) - f(x,\ y,\ z)}{l}$$

provided each of the above limits exists.

The above definition of partial differentiation can be extended to a function n variables, i.e., if $u = f(x_1, x_2, x_3, \ldots\ldots, x_n)$ be a function of n independent variables $x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_n$, then we

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can define $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \ldots\ldots\ldots\ldots\ldots, \frac{\partial u}{\partial x_n}$. In this case, $\frac{\partial u}{\partial x_1}$ means the first derivative of u w. r. t. x_1 only, treating the remaining variables $x_2, x_3, \ldots\ldots\ldots\ldots, x_n$ as constants, i.e.,

$$\frac{\partial u}{\partial x_1} = \lim_{h \to 0} \frac{f(x_1 + h,\ x_2\ ,\ x_3, \ldots\ldots\ldots\ldots, x_n) - f(x_1,\ x_2\ ,\ x_3, \ldots\ldots\ldots\ldots, x_n)}{h}$$

provided the above limit exists.

Similarly, $\frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \ldots\ldots\ldots\ldots\ldots, \frac{\partial u}{\partial x_n}$ can be defined.

2. Higher Order Partial Derivatives

Let $u = f(x, y)$ be a function of two independent variables x and y.

By definition, $\frac{\partial u}{\partial x}$ is the partial derivative of u w. r. t. x.

The partial derivative of $\frac{\partial u}{\partial x}$ w. r. t. x, if it exists, is called the second order partial derivative of u and this is denoted by $\frac{\partial^2 u}{\partial x^2}$ or u_{xx} i.e.,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(u_x) = (u_x)_x = u_{xx}.$$

Other second order partial derivatives of u are

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(u_y) = (u_y)_y = u_{yy},$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(u_x)_y = u_{xy},$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(u_y) = (u_y)_x = u_{yx}.$$

Obviously,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

Similarly, third order partial derivatives of u are

$$\frac{\partial^3 u}{\partial x^3}, \frac{\partial^3 u}{\partial y^3}, \frac{\partial^3 u}{\partial x^2 \partial y}, \frac{\partial^3 u}{\partial y^2 \partial x}.$$

Higher order partial derivatives of u can be formed as above.

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Example 1 : If $u = \sin\left(\sin\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

Solution : Given that

$$u = \sin\left(\sin\frac{y}{x}\right) \qquad ....(1)$$

Differentiating both sides of equation (1) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x} = \left\{\cos\left(\sin\frac{y}{x}\right)\right\}\left(\cos\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

This $\Rightarrow x\frac{\partial u}{\partial x} = -\left\{\cos\left(\sin\frac{y}{x}\right)\right\}\left(\cos\frac{y}{x}\right)\left(\frac{y}{x}\right) \qquad ....(2)$

Differentiating both sides of equation (1) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y} = \left\{\cos\left(\sin\frac{y}{x}\right)\right\}\left(\cos\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

This $\Rightarrow y\frac{\partial u}{\partial y} = \left\{\cos\left(\sin\frac{y}{x}\right)\right\}\left(\cos\frac{y}{x}\right)\left(\frac{y}{x}\right) \qquad ....(3)$

Adding equations (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Example 2 : If $u = \log(\tan x + \tan y + \tan z)$, prove that $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2.$

Solution : Given that

$$u = \log(\tan x + \tan y + \tan z) \qquad ....(1)$$

Differentiating both sides of equation (1) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}(\sec^2 x)$$

This $\Rightarrow \sin 2x\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}(\sec^2 x)(\sin 2x)$

$$= \frac{1}{\tan x + \tan y + \tan z}\left(\frac{1}{\cos^2 x}\right)(2\sin x\cos x)$$

$$\Rightarrow \sin 2x\frac{\partial u}{\partial x} = \frac{2\tan x}{\tan x + \tan y + \tan z} \qquad ....(2)$$

Now, differentiating both sides of equation (1) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z}(\sec^2 y)$$

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This $\Rightarrow \sin 2y \dfrac{\partial u}{\partial y} = \dfrac{1}{\tan x + \tan y + \tan z}(\sec^2 y)(\sin 2y)$

$$= \frac{1}{\tan x + \tan y + \tan z}\left(\frac{1}{\cos^2 y}\right)(2\sin y \cos y)$$

$$\Rightarrow \sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z} \qquad ....(3)$$

Similarly, differentiating both sides of equation (1) partially w. r. t. z, we get

$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z}(\sec^2 z)$$

This $\Rightarrow \sin 2z \dfrac{\partial u}{\partial z} = \dfrac{1}{\tan x + \tan y + \tan z}(\sec^2 z)(\sin 2z)$

$$= \frac{1}{\tan x + \tan y + \tan z}\left(\frac{1}{\cos^2 z}\right)(2\sin z \cos z)$$

$$\Rightarrow \sin 2z \frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z} \qquad ....(4)$$

Adding equations (2), (3) and (4), we get

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$$

Example 3 : If $x^3 + y^3 - x^3 y^2 z = 0,$ find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ when $x = y = 1.$

Solution : Given that

$$x^3 + y^3 - x^3 y^2 z = 0.$$

This $\Rightarrow x^3 y^2 z = x^3 + y^3$

$$\Rightarrow z = \frac{1}{y^2} + \frac{y}{x^3} \qquad ......(1)$$

Differentiating both sides of equation (1) partially w. r. t. x, we get

$$\frac{\partial z}{\partial x} = 0 + y\left(-\frac{3}{x^4}\right)$$

i.e., $\dfrac{\partial z}{\partial x} = -\dfrac{3y}{x^4}$

$\therefore \quad \dfrac{\partial z}{\partial x} = -3$, when $x = y = 1.$

Differentiating both sides of equation (1) partially w. r. t. y, we get

$$\frac{\partial z}{\partial y} = -\frac{2}{y^3} + \frac{1}{x^3}$$

$\therefore \quad \dfrac{\partial z}{\partial y} = -2 + 1 = -1$, when $x = y = 1.$

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Example 4 : If $u = e^{xyz}$, prove that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2 y^2 z^2)\, e^{xyz}$.

Solution : Given that

$$u = e^{xyz} \qquad ......(1)$$

Differentiating both sides of equation (1) partially w. r. t. z, we get

$$\frac{\partial u}{\partial z} = e^{xyz}(xy) \qquad ......(2)$$

Differentiating both sides of equation (2) partially w. r. t. y, we get

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = x\left\{e^{xyz}(1) + y\left(xze^{xyz}\right)\right\}$$

I.e., $$\frac{\partial^2 u}{\partial y \partial z} = x\,e^{xyz} + x^2 yz\, e^{xyz} \qquad .....(3)$$

Differentiating both sides of equation (3) partially w. r. t. x, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \left\{e^{xyz}(1) + x\left(yze^{xyz}\right)\right\} + yz\left\{e^{xyz}(2x) + x^2\left(yze^{xyz}\right)\right\}$$

I.e., $$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}\left\{(1 + xyz) + (2xyz + x^2 y^2 z^2)\right\}$$

$$= e^{xyz}(1 + 3xyz + x^2 y^2 z^2).$$

Example 5 : If $u = \log\left(\dfrac{x^2 + y^2}{xy}\right)$, prove that $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$.

Solution : Given that

$$u = \log\left(\frac{x^2 + y^2}{xy}\right).$$

This $\Rightarrow u = \log(x^2 + y^2) - \log x - \log y \qquad ....(1)$

Differentiating both sides of equation (1) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2}(2x) - \frac{1}{x} - 0$$

i.e., $$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} - \frac{1}{x} \qquad ......(2)$$

Differentiating both sides of equation (2) partially w. r. t. y, we get

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 2x\left\{-\frac{1}{(x^2 + y^2)^2}(2y)\right\} - 0$$

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i.e., $$\frac{\partial^2 u}{\partial y \partial x} = -\frac{4xy}{(x^2+y^2)^2} \qquad ....(3)$$

Now, differentiating both sides of equation (1) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}(2y) - 0 - \frac{1}{y}$$

i.e., $$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2} - \frac{1}{y} \qquad ......(4)$$

Differentiating both sides of equation (2) partially w. r. t. x, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 2y\left\{-\frac{1}{(x^2+y^2)^2}(2x)\right\} - 0$$

i.e., $$\frac{\partial^2 u}{\partial x \partial y} = -\frac{4xy}{(x^2+y^2)^2} \qquad .....(5)$$

From equations (3) and (5), we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

Example 6 : If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

Solution : Given that

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}} \qquad .....(1)$$

Differentiating both sides of equation (1) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}(2x)$$

i.e., $$\frac{\partial u}{\partial x} = -\frac{x}{\left\{\left(x^2+y^2+z^2\right)^{1/2}\right\}^3}$$

i.e., $$\frac{\partial u}{\partial x} = -xu^3 \qquad ......(2)$$

(using equation (1))

Again, differentiating both sides of equation (2) partially w. r. t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = -\left\{u^3(1) + x\left(3u^2\frac{\partial u}{\partial x}\right)\right\}$$

This $\Rightarrow \frac{\partial^2 u}{\partial x^2} = -u^3 - 3xu^2\frac{\partial u}{\partial x}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -u^3 - 3xu^2\left(-xu^3\right) \qquad \text{(using equation (2))}$$

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$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -u^3 + 3x^2u^5 \qquad ....(3)$$

Similarly, it can be proved that

$$\frac{\partial^2 u}{\partial y^2} = -u^3 + 3y^2u^5 \qquad ....(4)$$

and $$\frac{\partial^2 u}{\partial z^2} = -u^3 + 3z^2u^5 \qquad ....(5)$$

Adding equations (3), (4) and (5), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -3u^3 + 3u^5\,(x^2 + y^2 + z^2)$$

$$= -3u^3 + 3u^5\left(\frac{1}{u^2}\right) \qquad \text{(using equation (1))}$$

$$= -3u^3 + 3u^3$$

$$= 0.$$

Exercises

1. $u = a\sin\frac{x}{y} + \text{b}\cos\frac{y}{x}$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

2. If $u = x^m y^n$, prove that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

3. If $u = \tan^{-1}\frac{y}{x}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

4. If $u = (x^2 + y^2)^{-\frac{1}{2}}$, find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

Answers

1. 0

4. u^3

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