Matter Waves : de-Broglie Relation

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1 Wave Properties of Particles

Following the acceptance of the concept of duality for the radiation, de-Broglie (1924) proposed that moving objects also have wave as well as particle characteristics.

A photon of frequency ν and wavelength λ has the momentum

$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}$$

where, E is the energy of the photon, h is Planck's constant and c is the velocity of light in vacuum.

$\therefore$ Photon wavelength

$$\lambda = \frac{h}{p} \tag{1}$$

de-Broglie suggested that equation 1 is a general equation and applies to photons as well as material particles. Momentum of a particle of rest mass m_0 and velocity v is given by

$p = \gamma m_0 v$ and its de-Broglie wavelength is

$$\lambda = \frac{h}{\gamma m_0 v} \tag{2}$$

where $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ is the relativistic factor.

2 Velocity of de-Broglie Waves

Since de-Broglie waves are associated with moving particles, we would expect them to be moving with the speed of the particles.

Let the velocity of de Broglie waves is v_p. The velocity of waves, their frequency ν and wavelength λ are related by

$$v_p = \nu\lambda$$

$$\nu = \frac{E}{h} = \frac{\gamma m_0 c^2}{h}$$

$$\lambda = \frac{h}{\gamma m_0 v}$$

where, m_0 is the rest mass of the particle, v is the particle velocity.

$\therefore$ velocity of de Broglie wave is given by

$$v_p = \nu\lambda = \left(\frac{\gamma m_0 c^2}{h}\right)\left(\frac{h}{\gamma m_0 v}\right) = \frac{c^2}{v} \tag{3}$$

Because the particle velocity v must be smaller than c, the de-Broglie waves always travel faster than c and not with the velocity of the particle. This brings us in conflict with the theory of relativity. Another problem in the concept of de-Broglie wave is that particle is an entity with finite mass and dimensions tending to zero or we can say highly localised in space. However, a monochromatic wave is infinite in extent, i.e. completely incapable of being localised in a finite space.

We can think of a wave packet (i.e. a group of waves) associated with a particle. A wave packet can be made to have a resultant amplitude which is appreciably zero only in a very small region of space. Since a wave packet is made up of large number of waves, we can define a group velocity of the wave packet in addition to the velocities of the component waves, called their respective phase velocities. Representation of particles by a wave packet, removes the conflict in the wave velocity and particle velocity. The group velocity of the wave packet turns out to be equal to the velocity of the particle.

2.1 Phase and Group Velocity of de-Broglie Waves

Consider de-Broglie waves associated with a particle of rest mass m_0 travelling with a speed v. We can define angular frequency and wave number by

$$\omega = 2\pi\nu = \frac{2\pi E}{h} = \frac{2\pi\gamma m_0 c^2}{h} = \frac{2\pi m_0 c^2}{h\sqrt{1-\frac{v^2}{c^2}}}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{2\pi\gamma m_0 v}{h} = \frac{2\pi m_0 v}{h\sqrt{1-\frac{v^2}{c^2}}}$$

Phase velocity associated with de-Broglie waves is

$$v_p = \frac{\omega}{k} = \frac{c^2}{v}$$

Group velocity associated with de-Broglie waves is

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv} \tag{4}$$

$$\frac{d\omega}{dv} = \frac{d}{dv}\Big[\frac{2\pi m_0 c^2}{h}(1-\frac{v^2}{c^2})^{-\frac{1}{2}}\Big]$$
$$= \frac{2\pi m_0 c^2}{h}\Big[-\frac{1}{2}\big(1-\frac{v^2}{c^2}\big)^{-\frac{3}{2}}\big(-\frac{2v}{c}\big)\Big]$$
$$\frac{d\omega}{dv} = \frac{2\pi m_o v}{h(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \tag{5}$$

$$\frac{dk}{dv} = \frac{d}{dv}\Big[\frac{2\pi m_0 v}{h}\big(1-\frac{v^2}{c^2}\big)^{-1/2}\Big]$$
$$= \frac{2\pi m_0}{h}(1-\frac{v^2}{c^2})^{-1/2} + \frac{2\pi m_0 v}{h}\{-\frac{1}{2}(1-\frac{v^2}{c^2})^{-3/2}(-\frac{2v}{c^2})\}$$
$$= \frac{2\pi m_0}{h}(1-\frac{v^2}{c^2})^{-3/2}[1-\frac{v^2}{c^2}+\frac{v^2}{c^2}]$$
$$\frac{dk}{dv} = 2\pi m_0 h(1-v^2/c^2)^3/2 \tag{6}$$

using equation 4, 5 and 6, we have

$$v_g = \frac{d\omega/dv}{dk/dv} = \frac{2\pi m_0 v}{h(1-v^2/c^2)^3/2}\frac{h(1-v^2/c^2)^{3/2}}{2\pi m_o} = v \tag{7}$$

Therefore de-Broglie wave group associated with a body travels with the same velocity as the moving particle. The phase velocity of de-Broglie waves have no significance in itself. The wave picture of particle of momentum p is obtained by associating a wave group with it. The wave-group is formed by superposing wavelengths $(\lambda \pm n\Delta\lambda)$, where $n = 1, 2, 3, \ldots$ on a central wavelength $\lambda = \frac{h}{p}$.