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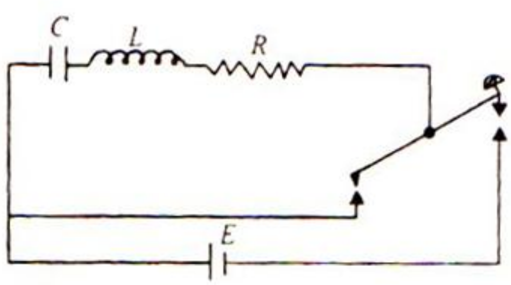
Expression for the growth or charging of an electric current in a circut cintaining inductance, capacitance and resistance (LCR) in series -

Consider a circuit containing an inductance L, a resistance R and a capacitance C in series connected to a battery of *e.m.f.* E, through a key. As soon as the key is pressed the capacitance starts getting charged through the inductance L and resistance R.

Let q be the charge on the capacitor after a time t. The charge on the capacitor slowly grows due to which there is current $i = \frac{dq}{dt}$ at any instant. The current changes at the rate $\frac{di}{dt} = \frac{d^2q}{dt^2}$.

$\therefore$ Potential difference across the capacitor $C = \frac{q}{C}$

Potential difference across the resistance



$$R = Ri = R\frac{dq}{dt}$$

Back *e.m.f.* in the inductance

$$L = -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$$

As the applied *e.m.f.* is E, we have

$$\frac{q}{C} + R\frac{dq}{dt} = E - L\frac{d^2q}{dt^2}$$

or $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \left(\frac{q}{C} - E\right) = 0$$

or $$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \left(\frac{q}{LC} - \frac{E}{L}\right) = 0$$

Put $$\frac{R}{L} = 2b \text{ and } \frac{1}{LC} = k^2$$

$\therefore$ $$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + \left(k^2q - \frac{E}{L}\right) = 0$$

or $$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + k^2\left(q - \frac{E}{k^2L}\right) = 0 \qquad ...(i)$$

Again put $q - \frac{E}{k^2L} = x$ so that $\frac{dx}{dt} = \frac{dq}{dt}$ and $\frac{d^2x}{dt^2} = \frac{d^2q}{dt^2}$.

Substituting in (*i*), we have

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2x = 0 \qquad ...(ii)$$

Let a trial solution of this differential equation be $x = e^{\alpha t}$, then

$$\frac{dx}{dt} = \alpha e^{\alpha t} \text{ and } \frac{d^2x}{dt^2} = \alpha^2 e^{\alpha t}$$

Substituting in Eq. (*ii*), we have

$$\alpha^2 e^{\alpha t} + 2b\alpha e^{\alpha t} + k^2 e^{\alpha t} = 0$$

or $$\alpha^2 + 2b\alpha + k^2 = 0$$

This is a quadratic equation in α. Its roots are

$$\alpha = -b \pm \sqrt{b^2 - k^2}$$

As there are two values of α, the general solution of differential equation (*ii*) is

$$x = Ae^{(-b + \sqrt{b^2 - k^2})t} + Be^{(-b - \sqrt{b^2 - k^2})t} \qquad ...(iii)$$

where A and B are arbitrary constants, the values of which can be determined from boundary conditions.

Now $$x = q - \frac{E}{k^2L} = q - EC$$

$\therefore$ $$q = x + EC = EC + Ae^{(-b + \sqrt{b^2 - k^2})t} + Be^{(-b - \sqrt{b^2 - k^2})t}$$

$EC = q_0$ the steady charge on the capacitor

$\therefore$ $$q = q_0 + Ae^{(-b + \sqrt{b^2 - k^2})t} + Be^{(-b - \sqrt{b^2 - k^2})t} \qquad ...(iv)$$

At $t = 0 \qquad q = 0$

$\therefore \quad q_0 + A + B = 0$ or $A + B = -q_0$

Differentiating Eq. (*iv*), we have $I = \frac{dq}{dt}$

$$= A(-b + \sqrt{b^2 - k^2})\, e^{(-b + \sqrt{b^2 - k^2})t} + B(-b\sqrt{b^2 - k^2})\, e^{(-b - \sqrt{b^2 - k^2})t} \qquad ...(v)$$

At $t = 0 \quad I = 0$

$$\therefore \quad 0 = A(-b + \sqrt{b^2 - k^2}) + B(-b - \sqrt{b^2 - k^2})$$

$$= -b(A + B) + (A - B)\sqrt{b^2 - k^2} = bq_0 + (A - B)\sqrt{b^2 - k^2}$$

$$\therefore \quad A - B = -\frac{bq_0}{\sqrt{b^2 - k^2}} \quad \text{Also} \quad A + B = -q_0$$

$$\therefore \quad A = \frac{-q_0}{2}\left(1 + \frac{b}{\sqrt{b^2 - k^2}}\right)$$

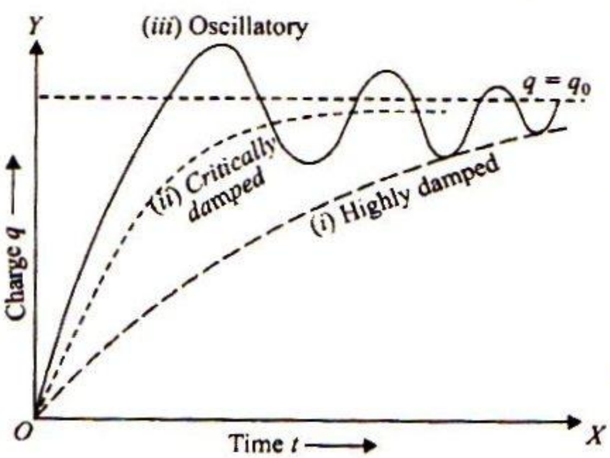
and

$$B = \frac{-q_0}{2}\left(1 - \frac{b}{\sqrt{b^2 - k^2}}\right)$$

Hence

$$q = q_0 - \frac{q_0}{2}\left(1 + \frac{b}{\sqrt{b^2 - k^2}}\right)e^{(-b + \sqrt{b^2 - k^2})t} - \frac{q_0}{2}\left(1 - \frac{b}{\sqrt{b^2 - k^2}}\right)e^{(-b - \sqrt{b^2 - k^2})t} \quad ...(vi)$$

Discussion. (*i*) When $b^2 > k^2$ *i.e.*, $\frac{R^2}{4L^2} > \frac{1}{LC}$ the quantity under the root sign is *positive* and therefore, co-efficient of t is *real*. The charge goes on increasing till it acquires the steady value q_0 [Fig. 10.23 (*i*)]. The charge is *dead beat i.e.*, highly damped.



(*ii*) When $b^2 = k^2$, $\frac{R^2}{4L^2} = \frac{1}{LC}$ the quantity under the root sign is *zero*. The charge is *critical one*, neither dead beat nor oscillatory. [Fig. 10.23 (*ii*)] *i.e.*, it is critically damped.

(*iii*) When $b^2 < k^2$ *i.e.*, $\frac{R^2}{4L^2}$, the quantity under the root sign is *negative* and $\sqrt{b^2 - k^2}$ is an imaginary quantity. Let it be equal to $j\omega$ where $j = \sqrt{-1}$ and $\omega = \sqrt{k^2 - b^2}$

Substituting in Eq. (*vi*), we have

$$q = q_0 - \frac{q_0}{2}\left(1 + \frac{b}{j\omega}\right)e^{(-b + j\omega)t} - \frac{q_0}{2}\left(1 - \frac{b}{j\omega}\right)e^{(-b - j\omega)t}$$

$$= q_0\left[1 - e^{-bt}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2}\right) + \frac{b}{j\omega}e^{-bt}\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right)\right]$$

$$= q_0 - q_0 e^{-bt}\left[\cos\omega t + \frac{b}{\omega}\sin\omega t\right]$$

$$= q_0 - q_0 e^{-bt}\frac{k}{\omega}\left[\frac{\omega}{k}\cos\omega t + \frac{b}{k}\sin\omega t\right]$$

Put $\frac{b}{k} = \sin\theta$ and $\frac{\omega}{k} = \cos\theta$ so that $\frac{b}{\omega} = \tan\theta$, then

$$q = q_0 - q_0 e^{-bt} \frac{k}{\omega}[\cos(\omega t - \theta)] \qquad \text{...(vii)}$$

Eq. (*vii*) represents a damped oscillatory charge, the charge being alternately greater and less than q_0 before settling down to the steady value q_0. [Fig. 10.23 (*iii*)]. The amplitude of vibration is

$$\frac{k}{\omega} e^{-bt} = \frac{k}{\sqrt{k^2 - b^2}} e^{-bt} = \frac{k}{\sqrt{k^2 - b^2}} e^{-\frac{R}{2L}t}$$

In circuits for which R is small the amplitude will die slowly. For $R = 0$, the amplitude is constant and the oscillations become simple harmonic. The maximum charge is much greater than the steady value q_0. It is possible that the maximum charge may raise the potential of the capacitor so high that the insulation may breakdown.

Current. The current in the circuit at any instant is obtained by differentiating the expression for charge, obtained in Eq. (*vii*).

$$I = \frac{dq}{dt} = q_0 k e^{-bt} \sin(\omega t - \theta) + q_0 \frac{kb}{\omega} e^{-bt} \cos(\omega t - \theta)$$

$$= q_0 e^{-bt} \frac{k^2}{\omega}\left[\frac{\omega}{k}\sin(\omega t - \theta) + \frac{b}{k}\cos(\omega t - \theta)\right]$$

$$= q_0 e^{-bt} \frac{k^2}{\omega}[\sin(\omega t - \theta)\cos\theta + \cos(\omega t - \theta)\sin\theta]$$

$$= q_0 e^{-bt} \frac{k^2}{\omega} \sin\omega t$$

Period of oscillation. The time period of oscillation is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k^2 - b^2}} = \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$$

If R is negligible, then $T = 2\pi\sqrt{LC}$

and then frequency $n = \frac{1}{T} = \frac{1}{2\pi\sqrt{LC}}$