Euler's Theorem

(For B.Sc./B.A. Part-I, Hons. And Subsidiary Courses of Mathematics)

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1. Homogeneous Function

A function f of two independent variables x, y is said to be a homogeneous function of degree n if it can be put in either of the following two forms :

$f(x,y) = x^n\,\phi\left(\frac{y}{x}\right)$, where ϕ denotes a function of $\frac{y}{x}$

or $\quad f(tx,ty) = t^n\, f(x,y)$, where t is any positive real number.

Similarly, a function f of three independent variables x, y, z is said to be a homogeneous function of degree n if it can be put in either of the following two forms :

$f(x,y,z) = x^n\,\phi\left(\frac{y}{x},\frac{z}{x}\right)$, where ϕ denotes a function of $\frac{y}{x}$ and $\frac{z}{x}$

or $\quad f(tx,ty,tz) = t^n\, f(x,y,z)$, where t is any positive real number.

The above definition can be extended to a function of any number of variables.

Example : The function

$$f(x,y) = \frac{x^4+y^4}{x-y}$$

is a homogeneous function of degree 3, since

$$f(x,y) = \frac{x^4+y^4}{x-y} = \frac{x^4\left[1+\left(\frac{y}{x}\right)^4\right]}{x\left[1-\left(\frac{y}{x}\right)\right]} = x^3\,\frac{\left[1+\left(\frac{y}{x}\right)^4\right]}{\left[1-\left(\frac{y}{x}\right)\right]} = x^3\phi\left(\frac{y}{x}\right)$$

or alternatively,

$$f(tx,ty) = \frac{(tx)^4+(ty^4)}{tx-ty} = \frac{t^4\left(x^4+y^4\right)}{t\left(x-y\right)} = t^3\,\frac{\left(x^4+y^4\right)}{\left(x-y\right)} = t^3 f\left(x,y\right).$$

Similarly, the function

$$f(x,y,z) = \frac{x^3+y^3+z^3}{x+y+z}$$

is a homogeneous function of degree 2, since

$$f(x,y,z) = \frac{x^3+y^3+z^3}{x+y+z} = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3+\left(\frac{z}{x}\right)^3\right]}{x\left[1+\left(\frac{y}{x}\right)+\left(\frac{z}{x}\right)\right]} = x^2\,\frac{\left[1+\left(\frac{y}{x}\right)^3+\left(\frac{z}{x}\right)^3\right]}{\left[1+\left(\frac{y}{x}\right)+\left(\frac{z}{x}\right)\right]} = x^2\phi\left(\frac{y}{x},\frac{z}{x}\right)$$

or alternatively,

$$f(tx,ty,tz)=\frac{(tx)^3+(ty^3)+(tz)^3}{tx+ty+tz}=\frac{t^3\left(x^3+y^3+z^3\right)}{t\left(x+y+z\right)}=t^2\,\frac{\left(x^3+y^3+z^3\right)}{\left(x+y+z\right)}=t^2 f\left(x,y,z\right).$$

Note: A polynomial function is a homogeneous function of degree n if all of its terms are of the same degree n.

Proof : Let f be a polynomial function in two independent variables $x, y,$ i.e.,

$$f(x,y)=a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+.........+a_{n-1}xy^{n-1}+a_ny^n.$$

Then $$f(x,y)=x^n\left[a_0+a_1\frac{y}{x}+a_2\frac{y^2}{x^2}+.........+a_{n-1}\frac{y^{n-1}}{x^{n-1}}+a_n\frac{y^n}{x^n}\right]$$

$$=x^n\left[a_0+a_1\left(\frac{y}{x}\right)+a_2\left(\frac{y}{x}\right)^2+.........+a_{n-1}\left(\frac{y}{x}\right)^{n-1}+a_n\left(\frac{y}{x}\right)^n\right]$$

$$=x^n\phi\left(\frac{y}{x}\right),$$ where ϕ is a function of $\frac{y}{x}$.

Example : The function

$$f(x,y)=x^5+6x^4y+7x^3y^2+2y^5$$

is a homogeneous function of degree 5, since

$$f(x,y)=x^5+6x^4y+7x^3y^2+2y^5$$

$$=x^5\left[1+6\left(\frac{y}{x}\right)+7\left(\frac{y}{x}\right)^2+2\left(\frac{y}{x}\right)^5\right]$$

$$=x^5\phi\left(\frac{y}{x}\right),$$ where ϕ is a function of $\frac{y}{x}$.

Similarly, the function

$$f(x,y,z)=x^4+3x^2y^2+4xyz^2+5yz^3+6y^4+7z^4$$

is a homogeneous function of degree 4, since

$$f(x,y,z)=x^4+3x^2y^2+4xyz^2+5yz^3+6y^4+7z^4$$

$$=x^4\left[1+3\left(\frac{y}{x}\right)^2+4\left(\frac{y}{x}\right)\left(\frac{z}{x}\right)^2+5\left(\frac{y}{x}\right)\left(\frac{z}{x}\right)^3+6\left(\frac{y}{x}\right)^4+7\left(\frac{z}{x}\right)^4\right]$$

$$=x^4\phi\left(\frac{y}{x},\frac{z}{x}\right),$$ where ϕ is a function of $\frac{y}{x}$ and $\frac{z}{x}$.

2. Euler's Theorem on Homogeneous Function of Two Variables

Statement : If u be a homogeneous function of degree n in two independent variables $x, y,$ then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

Proof : Let

$$u=A_1x^{\alpha_1}y^{\beta_1}+A_2x^{\alpha_2}y^{\beta_2}+A_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots\ldots\ldots+A_nx^{\alpha_n}y^{\beta_n} \qquad \ldots\ldots(1)$$

where $\alpha_1+\beta_1=\alpha_2+\beta_2=\alpha_3+\beta_3=\ldots\ldots\ldots\ldots\ldots=\alpha_n+\beta_n=n$

Differentiating both sides of equation (1) partially w. r. t. $x,$ we get

$$\frac{\partial u}{\partial x}=A_1\left(\alpha_1x^{\alpha_1-1}\right)y^{\beta_1}+A_2\left(\alpha_2x^{\alpha_2-1}\right)y^{\beta_2}+A_3\left(\alpha_3x^{\alpha_3-1}\right)y^{\beta_3}+\ldots\ldots+A_n\left(\alpha_nx^{\alpha_n-1}\right)y^{\beta_n}$$

This $\Rightarrow x\frac{\partial u}{\partial x}=A_1\alpha_1x^{\alpha_1}y^{\beta_1}+A_2\alpha_2x^{\alpha_2}y^{\beta_2}+A_3\alpha_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots+A_n\alpha_nx^{\alpha_n}y^{\beta_n} \qquad \ldots\ldots(2)$

Now, differentiating both sides of equation (1) partially w. r. t. $y,$ we get

$$\frac{\partial u}{\partial y}=A_1x^{\alpha_1}\left(\beta_1y^{\beta_1-1}\right)+A_2x^{\alpha_2}\left(\beta_2y^{\beta_2-1}\right)+A_3x^{\alpha_3}\left(\beta_3y^{\beta_3-1}\right)+\ldots\ldots+A_nx^{\alpha_n}\left(\beta_ny^{\beta_n-1}\right)$$

This $\Rightarrow y\frac{\partial u}{\partial y}=A_1\beta_1x^{\alpha_1}y^{\beta_1}+A_2\beta_2x^{\alpha_2}y^{\beta_2}+A_3\beta_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots.+A_n\beta_nx^{\alpha_n}y^{\beta_n} \qquad \ldots\ldots(3)$

Adding equations (2) and (3), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(\alpha_1+\beta_1)A_1x^{\alpha_1}y^{\beta_1}+(\alpha_2+\beta_2)A_2x^{\alpha_2}y^{\beta_2}+(\alpha_3+\beta_3)A_3x^{\alpha_3}y^{\beta_3}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots+(\alpha_n+\beta_n)A_nx^{\alpha_n}y^{\beta_n}$$

$$=nA_1x^{\alpha_1}y^{\beta_1}+nA_2x^{\alpha_2}y^{\beta_2}+nA_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots+nA_nx^{\alpha_n}y^{\beta_n}$$

$$(\because\ \alpha_1+\beta_1=\alpha_2+\beta_2=\alpha_3+\beta_3=\ldots\ldots\ldots\ldots\ldots=\alpha_n+\beta_n=n)$$

$$=n\left(A_1x^{\alpha_1}y^{\beta_1}+A_2x^{\alpha_2}y^{\beta_2}+A_3x^{\alpha_3}y^{\beta_3}+\ldots\ldots\ldots\ldots\ldots+A_nx^{\alpha_n}y^{\beta_n}\right)$$

$=nu$ (using equation (1))

i.e., $$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu}$$

Corollary : If u be a homogeneous function of degree n in two independent variables $x, y,$ then

(i) $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$

(ii) $x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$

(iii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$

Proof : (i) Since u is a homogeneous function of degree n in two independent variables $x, y,$ therefore, by Euler's Theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \qquad ......(1)$$

Differentiating both sides of equation (1) partially w. r. t. $x,$ we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}(1)+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$$

This $\Rightarrow x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$ ....(2)

Hence (i) is proved.

(ii) Differentiating both sides of equation (1) partially w. r. t. $y,$ we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}(1)=n\frac{\partial u}{\partial y}$$

This $\Rightarrow x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y}$ $\left(\because \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}\right)$

$\Rightarrow x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$ ....(3)

Hence (ii) is proved.

(iii) Multiplying equations (2) and (3) by x and y respectively and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=(n-1)(nu) \quad \text{(using equation (1))}$$

i.e., $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$

This proves (iii).

3. Euler's Theorem on Homogeneous Function of Three Variables

Statement : If u be a homogeneous function of degree n in three independent variables $x, y, z,$ then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$

Proof : Let

$$u = A_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \ldots\ldots\ldots + A_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \qquad \ldots\ldots(1)$$

where $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = \alpha_3 + \beta_3 + \gamma_3 = \ldots\ldots\ldots = \alpha_n + \beta_n + \gamma_n = n$

Differentiating both sides of equation (1) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x} = A_1\left(\alpha_1 x^{\alpha_1 - 1}\right) y^{\beta_1} z^{\gamma_1} + A_2\left(\alpha_2 x^{\alpha_2 - 1}\right) y^{\beta_2} z^{\gamma_2} + A_3\left(\alpha_3 x^{\alpha_3 - 1}\right) y^{\beta_3} z^{\gamma_3}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots + A_n\left(\alpha_n x^{\alpha_n - 1}\right) y^{\beta_n} z^{\gamma_n}$$

This $\Rightarrow$

$$\left.\begin{aligned} x\frac{\partial u}{\partial x} &= A_1\alpha_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2\alpha_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3\alpha_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} \\ &+ \ldots\ldots\ldots\ldots + A_n\alpha_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \end{aligned}\right\} \qquad \ldots\ldots.(2)$$

Now, differentiating both sides of equation (1) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y} = A_1 x^{\alpha_1}\left(\beta_1 y^{\beta_1 - 1}\right) z^{\gamma_1} + A_2 x^{\alpha_2}\left(\beta_2 y^{\beta_2 - 1}\right) z^{\gamma_2} + A_3 x^{\alpha_3}\left(\beta_3 y^{\beta_3 - 1}\right) z^{\gamma_3}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots + A_n x^{\alpha_n}\left(\beta_n y^{\beta_n - 1}\right) z^{\gamma_n}$$

This $\Rightarrow$

$$\left.\begin{aligned} y\frac{\partial u}{\partial y} &= A_1\,\beta_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2\,\beta_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3\,\beta_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} \\ &+ \ldots\ldots\ldots\ldots + A_n\,\beta_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \end{aligned}\right\} \qquad \ldots\ldots.(3)$$

Similarly, differentiating both sides of equation (1) partially w. r. t. z, we get

$$\frac{\partial u}{\partial z} = A_1 x^{\alpha_1} y^{\beta_1}\left(\gamma_1 z^{\gamma_1 - 1}\right) + A_2 x^{\alpha_2} y^{\beta_2}\left(\gamma_2 z^{\gamma_2 - 1}\right) + A_3 x^{\alpha_3} y^{\beta_3}\left(\gamma_3 z^{\gamma_3 - 1}\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots + A_n x^{\alpha_n} y^{\beta_n}\left(\gamma_n z^{\gamma_n - 1}\right)$$

This $\Rightarrow$

$$\left.\begin{aligned} z\frac{\partial u}{\partial z} &= A_1\,\gamma_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2\,\gamma_2\, x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3\,\gamma_3\, x^{\alpha_3} y^{\beta_3} z^{\gamma_3} \\ &+ \ldots\ldots\ldots\ldots + A_n\,\gamma_n\, x^{\alpha_n} y^{\beta_n} z^{\gamma_n} \end{aligned}\right\} \qquad \ldots\ldots.(4)$$

Adding equations (2), (3) and (4), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = (\alpha_1 + \beta_1 + \gamma_1) A_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + (\alpha_2 + \beta_2 + \gamma_2) A_2\, x^{\alpha_2} y^{\beta_2} z^{\gamma_2} +$$

$$(\alpha_3 + \beta_3 + \gamma_3) A_3\, x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \ldots\ldots + (\alpha_n + \beta_n + \gamma_n) A_n\, x^{\alpha_n} y^{\beta_n} z^{\gamma_n}$$

$$= n\,A_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + n\,A_2\, x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + n\,A_3\, x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \ldots\ldots\ldots +$$

$$n\,A_n\, x^{\alpha_n} y^{\beta_n} z^{\gamma_n}$$

$$(\because \alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = \ldots\ldots = \alpha_n + \beta_n + \gamma_n = n)$$

$$= n\left(A_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \ldots\ldots + A_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n}\right)$$

$= n\,u$ (using equation (1))

i.e.,

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\,u}$$

Example 1 : Verify Euler's Theorem when $u=\dfrac{x(x^3-y^3)}{x^3+y^3}$.

Solution : According to Euler's Theorem, if u be a homogeneous function of degree n in two independent variables $x, y,$ then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

Given that

$$u=\frac{x(x^3-y^3)}{x^3+y^3} \qquad .....(1)$$

i.e.,
$$u=\frac{x^4\left[1-\left(\frac{y}{x}\right)^3\right]}{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}=x\frac{\left[1-\left(\frac{y}{x}\right)^3\right]}{\left[1+\left(\frac{y}{x}\right)^3\right]}=x\phi\left(\frac{y}{x}\right),$$ where ϕ is a function of $\dfrac{y}{x}$.

This $\Rightarrow$ The given function u is a homogeneous function of degree 1 in two independent variables $x, y.$ Therefore Euler's Theorem will be verified if we can prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u.$$

Taking logarithm of both sides of equation (1), we get

$$\log u=\log x+\log(x^3-y^3)-\log(x^3+y^3) \qquad ....(2)$$

Now, differentiating both sides of equation (2) partially w. r. t. $x,$ we get

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{1}{x}+\frac{1}{x^3-y^3}(3x^2)-\frac{1}{x^3+y^3}(3x^2)$$

This $\Rightarrow \dfrac{1}{u}\left(x\dfrac{\partial u}{\partial x}\right)=1+\dfrac{3x^3}{x^3-y^3}-\dfrac{3x^3}{x^3+y^3}$ ....(3)

Similarly, differentiating both sides of equation (2) partially w. r. t. $y,$ we get

$$\frac{1}{u}\frac{\partial u}{\partial y}=0+\frac{1}{x^3-y^3}(-3y^2)-\frac{1}{x^3+y^3}(3y^2)$$

This $\Rightarrow \dfrac{1}{u}\left(y\dfrac{\partial u}{\partial y}\right)=-\dfrac{3y^3}{x^3-y^3}-\dfrac{3y^3}{x^3+y^3}$ ....(4)

Adding equations (3) and (4), we get

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=1+\frac{3(x^3-y^3)}{x^3-y^3}-\frac{3(x^3+y^3)}{x^3+y^3}$$
$$=1+3-3$$
$$=1$$

This $\Rightarrow x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=u$

$\Rightarrow$ Euler's Theorem is verified for the given function.

Example 2 : Verify Euler's Theorem when $u=\dfrac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}$.

Solution : According to Euler's Theorem, if u be a homogeneous function of degree n in two independent variables $x, y,$ then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

Given that

$$u=\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}} \qquad .....(1)$$

i.e.,
$$u=\frac{x^{\frac{1}{4}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{4}}\right]}{x^{\frac{1}{5}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{5}}\right]}=x^{\frac{1}{20}}\frac{\left[1+\left(\frac{y}{x}\right)^{\frac{1}{4}}\right]}{\left[1+\left(\frac{y}{x}\right)^{\frac{1}{5}}\right]}=x^{\frac{1}{20}}\phi\left(\frac{y}{x}\right),$$ where ϕ is a function of $\frac{y}{x}$.

This $\Rightarrow$ The given function u is a homogeneous function of degree $\frac{1}{20}$ in two independent variables $x, y.$ Therefore Euler's Theorem will be verified if we can prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{20}u.$$

Taking logarithm of both sides of equation (1), we get

$$\log u=\log\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)-\log\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right) \qquad ....(2)$$

Now, differentiating both sides of equation (2) partially w. r. t. $x,$ we get

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{1}{x^{\frac{1}{4}}+y^{\frac{1}{4}}}\left(\frac{1}{4}x^{-\frac{3}{4}}\right)-\frac{1}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}\left(\frac{1}{5}x^{-\frac{4}{5}}\right)$$

This $\Rightarrow \dfrac{1}{u}\left(x\dfrac{\partial u}{\partial x}\right)=\dfrac{x^{\frac{1}{4}}}{4\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)}-\dfrac{x^{\frac{1}{5}}}{5\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)}$ ....(3)

Similarly, differentiating both sides of equation (2) partially w. r. t. $y,$ we get

$$\frac{1}{u}\frac{\partial u}{\partial y}=\frac{1}{x^{\frac{1}{4}}+y^{\frac{1}{4}}}\left(\frac{1}{4}y^{-\frac{3}{4}}\right)-\frac{1}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}\left(\frac{1}{5}y^{-\frac{4}{5}}\right)$$

This $\Rightarrow \dfrac{1}{u}\left(y\dfrac{\partial u}{\partial y}\right)=\dfrac{y^{\frac{1}{4}}}{4\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)}-\dfrac{y^{\frac{1}{5}}}{5\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)}$ ....(4)

Adding equations (3) and (4), we get

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{4\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)}-\frac{x^{\frac{1}{5}}+y^{\frac{1}{5}}}{5\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)}$$

$$=\frac{1}{4}-\frac{1}{5}$$

$$=\frac{1}{20}$$

This $\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{20}u$

$\Rightarrow$ Euler's Theorem is verified for the given function.

Example 3 : If $u=\tan^{-1}\frac{x^2+y^2}{x+y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u.$

Solution : Given that

$$u=\tan^{-1}\frac{x^2+y^2}{x+y}.$$

This $\Rightarrow \tan u=\frac{x^2+y^2}{x+y}=\frac{x^2\left[1++\left(\frac{y}{x}\right)^2\right]}{x\left[1++\frac{y}{x}\right]}=x\frac{\left[1++\left(\frac{y}{x}\right)^2\right]}{\left[1++\frac{y}{x}\right]}=x\,\varphi\left(\frac{y}{x}\right)$, where ϕ is a function of $\frac{y}{x}$.

$\Rightarrow \tan u$ is a homogeneous function of degree 1 in two independent variables x, y.

Let $v=\tan u$ ......(1)

Then v is a homogeneous function of degree 1 in two independent variables $x, y.$ Therefore, by Euler's Theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=v.$$

This $\Rightarrow x\left(\sec^2 u\frac{\partial u}{\partial x}\right)+y\left(\sec^2 u\frac{\partial u}{\partial y}\right)=\tan u$

$$\left(\because \frac{\partial v}{\partial x}=\sec^2 u\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y}=\sec^2 u\frac{\partial u}{\partial y}\text{ and } v=\tan u, \text{ by equation (1)}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\tan u}{\sec^2 u}$$

$$=\frac{\sin u}{\cos u}\left(\cos^2 u\right)$$

$$=\sin u\cos u$$

$$=\frac{1}{2}\left(2\sin u\cos u\right)$$

$$=\frac{1}{2}\sin 2u.$$

Example 4 : If $u=\cos^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+\dfrac{1}{2}\cot u=0.$

Solution : Given that

$$u=\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}.$$

This $\Rightarrow \cos u=\dfrac{x+y}{\sqrt{x}+\sqrt{y}}=\dfrac{x\left[1++\dfrac{y}{x}\right]}{\sqrt{x}\left[1++\sqrt{\dfrac{y}{x}}\right]}=x^{\frac{1}{2}}\dfrac{\left[1++\dfrac{y}{x}\right]}{\left[1++\sqrt{\dfrac{y}{x}}\right]}=x^{\frac{1}{2}}\varphi\left(\dfrac{y}{x}\right)$, where ϕ is a function of $\dfrac{y}{x}$.

$\Rightarrow \cos u$ is a homogeneous function of degree $\dfrac{1}{2}$ in two independent variables x, y.

Let $v=\cos u$ ......(1)

Then v is a homogeneous function of degree $\dfrac{1}{2}$ in two independent variables $x, y.$ Therefore, by Euler's Theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{1}{2}v.$$

This $\Rightarrow x\left(-\sin u\dfrac{\partial u}{\partial x}\right)+y\left(-\sin u\dfrac{\partial u}{\partial y}\right)=\dfrac{1}{2}\cos u$

$$\left(\because \frac{\partial v}{\partial x}=-\sin u\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y}=-\sin u\frac{\partial u}{\partial y}\text{ and } v=\cos u, \text{ by equation (1)}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\frac{\cos u}{\sin u}$$

$$=-\frac{1}{2}\cot u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0.$$

Example 5 : If $u=\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0.$

Solution : Given that

$$u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right) \qquad ......(1)$$

Let $\sin^{-1}\left(\dfrac{x}{y}\right)=\theta.$

Then $\sin\theta=\dfrac{x}{y}.$

This $\Rightarrow \tan\theta=\dfrac{x}{\sqrt{y^2-x^2}}$

$$\Rightarrow \theta = \tan^{-1}\frac{x}{\sqrt{y^2 - x^2}}$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\frac{x}{\sqrt{y^2 - x^2}} \qquad .....(2)$$

Substituting the value of $\sin^{-1}\left(\frac{x}{y}\right)$ from equation (2) in equation (1), we get

$$u = \tan^{-1}\frac{x}{\sqrt{y^2 - x^2}} + \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\frac{\frac{x}{\sqrt{y^2 - x^2}} + \frac{y}{x}}{1 - \left(\frac{x}{\sqrt{y^2 - x^2}}\right)\left(\frac{y}{x}\right)}$$

$$= \tan^{-1}\frac{x^2 + y\sqrt{y^2 - x^2}}{x\sqrt{y^2 - x^2} - xy}.$$

This $\Rightarrow \tan u = \dfrac{x^2 + y\sqrt{y^2 - x^2}}{x\sqrt{y^2 - x^2} - xy} = \dfrac{x^2\left[1 + \frac{y}{x}\sqrt{\left(\frac{y}{x}\right)^2 - 1}\right]}{x^2\left[\sqrt{\left(\frac{y}{x}\right)^2 - 1} - \frac{y}{x}\right]} = \dfrac{\left[1 + \frac{y}{x}\sqrt{\left(\frac{y}{x}\right)^2 - 1}\right]}{\left[\sqrt{\left(\frac{y}{x}\right)^2 - 1} - \frac{y}{x}\right]} = x^0\,\varphi\left(\frac{y}{x}\right),$

where ϕ is a function of $\frac{y}{x}$.

$\Rightarrow \tan u$ is a homogeneous function of degree zero in two independent variables x, y.

Let $v = \tan u$ ......(3)

Then v is a homogeneous function of degree zero in two independent variables x, y. Therefore, by Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0.$$

This $\Rightarrow x\left(\sec^2 u\,\frac{\partial u}{\partial x}\right) + y\left(\sec^2 u\,\frac{\partial u}{\partial y}\right) = 0$

$$\left(\because \frac{\partial v}{\partial x} = \sec^2 u\,\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y} = \sec^2 u\,\frac{\partial u}{\partial y} \text{ and } v = \tan u, \text{ by equation (3)}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{0}{\sec^2 u}$$

$$= 0.$$

Example 6 : If $u = \sin\left(\sqrt{x}+\sqrt{y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\cos\left(\sqrt{x}+\sqrt{y}\right).$

Solution : Given that

$$u = \sin\left(\sqrt{x}+\sqrt{y}\right) \qquad \text{......(1)}$$

This $\Rightarrow \sin^{-1} u = \sqrt{x}+\sqrt{y} = \sqrt{x}\left[1++\sqrt{\dfrac{y}{x}}\right] = x^{\frac{1}{2}}\varphi\left(\dfrac{y}{x}\right)$, where ϕ is a function of $\dfrac{y}{x}$.

$\Rightarrow \sin^{-1} u$ is a homogeneous function of degree $\dfrac{1}{2}$ in two independent variables x, y.

Let $v = \sin^{-1} u$ ......(2)

Then v is a homogeneous function of degree $\dfrac{1}{2}$ in two independent variables $x, y.$ Therefore, by Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{1}{2}v.$$

This $\Rightarrow x\left(\dfrac{1}{\sqrt{1-u^2}}\dfrac{\partial u}{\partial x}\right) + y\left(\dfrac{1}{\sqrt{1-u^2}}\dfrac{\partial u}{\partial y}\right) = \dfrac{1}{2}\sin^{-1} u$

$$\left(\because \frac{\partial v}{\partial x} = \frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y} = \frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial y} \text{ and } v = \sin^{-1} u, \text{ by equation (2)}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\left(\sin^{-1} u\right)\left(\sqrt{1-u^2}\right)$$

$$= \frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{1-\sin^2(\sqrt{x}+\sqrt{y})}\right) \quad \text{(using equation (1))}$$

$$= \frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\cos\left(\sqrt{x}+\sqrt{y}\right).$$

Exercises

1. Verify Euler's Theorem when $u = x^3 \log\dfrac{y}{x}$.

2. If $u = \sin\sqrt{\dfrac{x-y}{x+y}}$, prove that Euler's formula $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$ holds good.

3. If $u = \sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, prove that $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\dfrac{\partial u}{\partial y}$.

4. If $\sin u = \dfrac{x^3+y^3}{x+y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\tan u.$