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Adiabatic Expansion of Radiation: Wein's Displacement Law

Radiation emitted by a body is spread over a continuous spectrum. Wein in 1893 showed that $E_\lambda d\lambda$, i.e. amount of energy contained in the spectral region included within wavelength λ and λ+dλ emitted by a blackbody at temperature T is of the form

$$E_\lambda d\lambda = \frac{A}{\lambda^5} f(\lambda T) d\lambda \qquad (1)$$

where A is a constant and f(λT) is a function of the product (λT).

Consider an imaginary experiment performed with a spherical enclosure of volume V having perfectly reflecting walls which are made of elastic material and capable of slowly moving outwards. Let the enclosure is maintained at some temperature T and let a small piece of black body of negligible heat capacity be placed in it. Radiations inside the enclosure comes into equilibrium with the black body very soon. Now, the black body is taken out so that the enclosure remains filled with diffuse radiation at T.

Now we allow the black radiation to expand adiabatically. We may suppose that the wall begins to move outward with uniform velocity v(v<<c) slowly. When the volume has expanded to V' the temperature will fall to say T' and the quality of radiation may change but remains black radiation characteristic of lower temperature T'.

Let us now calculate the change in frequency due to Doppler effect which every wave will suffer on reflection at a moving wall. The waves inside the walls will be incident on the walls at all angles. In case of normal incidence, a wave of frequency ν will be changed according to Doppler principle to $\nu\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$ on

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reaching the wall. The frequency of reflected wave is again decreased in the ratio $\nu\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$.

The resultant frequency is now $\nu + d\nu = \nu\frac{1-\frac{v}{c}}{1+\frac{v}{c}} = \nu(1-\frac{2v}{c})$

$\because$ v/c is very small $\frac{d\nu}{\nu} = -\frac{2v}{c}\ or\ \frac{d\lambda}{\lambda} = \frac{2v}{c}$

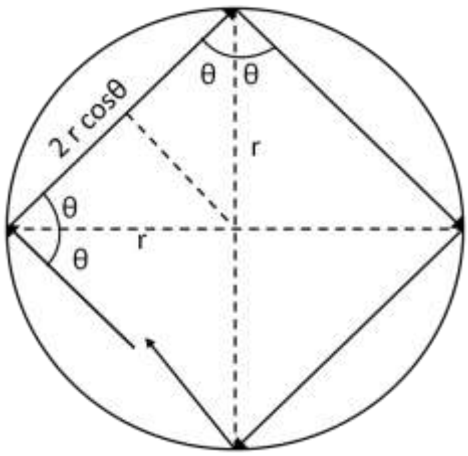
For oblique incidence, so that radiation makes an angle θ with normal to the surface, the effective part of v is that perpendicular to the wavefront i.e. $vcos\theta$

$$d\lambda = \frac{2vcos\theta}{c}\lambda$$

dλ is change in wavelength in one reflection.

Between successive reflections the wave travels a distance $2rcos\theta$ and hence the number of reflections per second is $\frac{c}{2rcos\theta}$

$\because$ wavelength change per second is



$d\lambda = \left(\frac{c}{2rcos\theta}\right)\left(\frac{2vcos\theta}{c}\right)\lambda$

$d\lambda = \frac{v\lambda}{r}$

$\frac{d\lambda}{\lambda} = \frac{v}{r} = \frac{\delta r}{r}$

$\because v = \delta r$ is the distance travelled in 1 s.

The temperature of the enclosure is also changed.

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Since the expansion is adiabatic, a part of energy is spent in doing work.

$$\delta Q = dU + PdV = 0$$

dU is change in internal energy, PdV is work done by the radiant energy.

U = uV , where, u is density of radiation and Vis volume of sphere, P = u/3

$$d(uV) + \frac{u}{3}dV = 0$$

$$Vdu + udV + \frac{u}{3}dV = 0$$

$$\frac{du}{u} + \frac{4}{3}dV = 0$$

$$uV^{\frac{4}{3}} = const$$

This can also be written as $PV^{\frac{4}{3}} = constant$

Thus radiation behaves like a gas having its $\gamma = \frac{4}{3}$.

Now, from Stefan's law we have, u= AT^4 and $V = \frac{4}{3}\pi r^3$

$\therefore T^4 r^4 = constant$ or r T = constant

$\frac{dT}{T} = -\frac{\delta r}{r}$

$\frac{d\lambda}{\lambda} = -\frac{dT}{T}$

$\boldsymbol{\lambda T = constant}$ This is Wein's Displacement law.

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We will now prove $u_\lambda \lambda^5 =$ constant

Suppose we isolate wavelengths lying between λ and λ+d λ in spherical chamber and subject these alone to adiabatic expansion. The work done in adiabatic expansion is $\frac{1}{3} u_\lambda d\lambda.\Delta V$ and this should be equal to decrease in total energy.

i.e $-\Delta(u_\lambda d\lambda V)$

$\therefore \frac{1}{3} u_\lambda d\lambda\Delta V = -\Delta(u_\lambda d\lambda V)$

d λ changes in the same way as λ, $\frac{\Delta d\lambda}{d\lambda} = \frac{\Delta\lambda}{\lambda}, V \propto \lambda^3$

$$\frac{1}{3} u_\lambda d\lambda\Delta V = -\Delta u_\lambda d\lambda V - u_\lambda \Delta d\lambda V - u_\lambda d\lambda\Delta V$$

Dividing by $u_\lambda d\lambda V$, we have

$$\frac{1}{3}\frac{\Delta V}{V} = -\frac{\Delta u_\lambda}{u_\lambda} - \frac{\Delta d\lambda}{d\lambda} - \frac{\Delta V}{V}$$

$$\frac{1}{3}.\frac{3\Delta\lambda}{\lambda} = -\frac{\Delta u_\lambda}{u_\lambda} - \frac{\Delta\lambda}{\lambda} - \frac{3\Delta\lambda}{\lambda}$$

$$\frac{5\Delta\lambda}{\lambda} + \frac{\Delta u_\lambda}{u_\lambda} = 0$$

$$V \propto \lambda^3$$

$$\Delta V = 3\lambda^2\Delta\lambda$$

$$\frac{\Delta V}{V} = \frac{3\lambda^2\Delta\lambda}{\lambda^3} = 3\frac{\Delta\lambda}{\lambda}$$

$$u_\lambda.\lambda^5 = \text{constant} = u'_\lambda \lambda'^5$$

Where, u'_λ is the density of radiation λ′ to which λ has been transformed by expansion. This will correspond to equilibrium radiation density at temperature $T' = \frac{\lambda T}{\lambda'}$.

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$\therefore u_\lambda$ must be a function of T, hence the constant must involve T. But the constant is such that it remains constant throughout suchadiabatic alteration of wavelength.

λ T is a constant for this process.

$\therefore$ $\boldsymbol{u_\lambda d\lambda = \frac{A}{\lambda^5} f(\lambda T) d\lambda}$ Wein's Distribution Law

The above equation can be written in other form

$$u_\lambda d\lambda = AT^5 F(\lambda T) d\lambda$$

Where $F(\lambda T) = (\lambda T^{-5}) f(\lambda T)$