Waves and Vibrations

Dr. Priti Mishra

1 WAVES

1.1 INTRODUCTION

Waves arise in a wide range of physical phenomena. They occur as ripples on a pond and as seismic waves following an earthquake. Music is carried by sound waves and most of what we know about the Universe comes from electromagnetic waves that reach the Earth. Furthermore, we communicate with each other through a variety of different waves. At the microscopic level, the particles of matter have a wave nature as expressed by quantum wave mechanics. At the other end of the scale, scientists are trying to detect gravitational waves that are predicted to occur when massive astronomical objects like black holes move rapidly. Even a Mexican wave travelling around a sports arena has many of the characteristics of wave motion. It is not surprising therefore that waves are at the heart of many branches of the physical sciences including optics, electromagnetism, quantum mechanics and acoustics.

A wave is a disturbance that travels through a medium and transports energy and momentum without *the transport of particles of the medium.*
Examples: The ripples on a pond, the sound we hear, visible light, radio and TV signals are a few examples of waves. Sound, light and radio waves provide us with an effective means of transmitting and receiving energy and information.

Waves are of two types :

1. Mechanical waves and

2. Electromagnetic waves.

Mechanical waves require material medium for their propagation. Elasticity and density are the key properties of the material medium that are required for the propagation of mechanical waves. That is why the mechanical waves are sometimes referred to as elastic waves.

Example: A sound wave, the disturbance is pressure-variation in a medium.

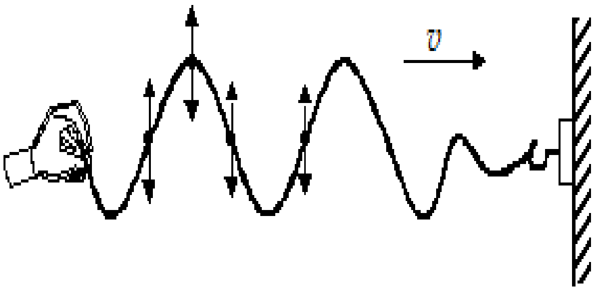

Figure 1: Transverse waves moving along a rope

In a mechanical wave motion, it is the disturbance that moves and not the particles of the medium.

Electromagnetic waves require absolutely no material medium for their propagation. They can travel through vacuum. Light, TV signals, radio waves, X-rays, etc. are examples of non mechanical waves. These are electromagnetic in nature. In an electromagnetic wave, energy travels in the form of electric and magnetic fields. **Example:** In the transmission of light in a medium or vacuum, the disturbance is the variation of the strengths of the electric and magnetic fields.

2 Classification of Waves

Based on the direction of motion of particles with respect to the direction of propagation mechanical waves are classified into the following two categories:

1. **Transverse Waves :** In such waves, the oscillatory motion of the particles of the medium is perpendicular to the direction of propagation. Consider the wave travelling along a rope. The direction of propagation of the wave is along the rope, but the individual particles of the rope vibrate up and down. The electromagnetic wave (light, radio waves, X-rays, etc.) through not mechanical, are said to be transverse, as the electric and magnetic field vibrate in direction perpendicular to the direction of propagation.

2. **Longitudinal Waves** : In these waves, the direction of vibration of the particles of the medium is parallel to the direction of propagation.

The figure 2 shows a long and elastic spring. When we repeatedly push and pull on end of the spring, the compression and rarefaction of the spring travel along the spring. A particle on the spring moves back and forth, parallel and anti-parallel to the direction of the wave velocity. Sound waves

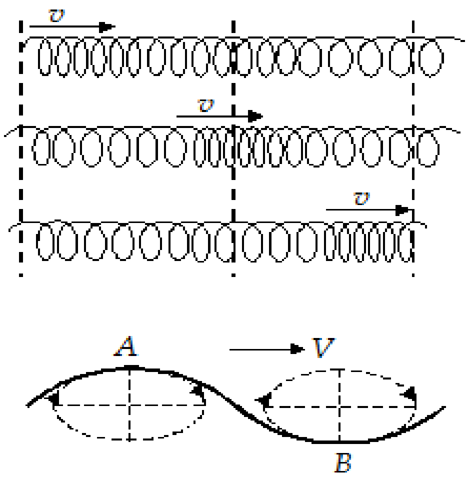

Figure 2: Longitudinal wave

in air are also longitudinal. Some waves (for example,ripples on the surface of a pond) are neither transverse nor longitudinal but a combination of the two. The particles of the medium vibrate up and down, and back and forth simultaneously describing ellipses in a vertical plane. In strings, mechanical waves are always transverse, when the string is under a tension. In gases and liquids, mechanical waves are always longitudinal, e.g., sound waves in air or water. This is because fluids cannot sustain shear. They do not posses rigidity. They posses volume elasticity, because of which the variations of pressure (i.e., compression and rarefaction) can travel through them. For this reason, the longitudinal waves are also called *pressure waves.*

The waves on the surface of water are of two kinds: Capillary waves and gravity waves. Capillary waves are ripples of fairly short wavelength - no more than a few centimeters. The restoring force that produces these waves is the surface tension of water. Gravity waves have wavelength of several meter and restoring force is the pull of gravity.

In solids, mechanical waves (may be sound) can be either transverse or longitudinal depending on the mode of excitation. The speeds of the two waves in the same solid are different (longitudinal waves travel faster than transverse waves).

A gas can sustain only longitudinal waves because transverse waves require a shear force to maintain them. Both transverse and longitudinal waves can travel in a solid.

Based on the extent of the medium waves can further be classified into *progressive or traveling waves and standing waves.*

One of the simplest ways to demonstrate progressive or traveling wave

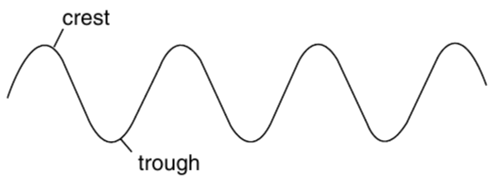

Figure 3: Progressive transverse waves moving along a string

motion is to take the loose end of a long rope which is fixed at the other end and to move the loose end quickly up and down. Crests and troughs of the waves move down the rope, and if the rope were infinitely long such waves would be called *progressive waves or traveling waves–these are waves traveling in an* <u>*unbounded*</u> *medium free from possible reflection* (Figure 3). If the medium is <u>limited</u> in extent; for example, if the rope were reduced to a violin string, fixed at both ends, the progressive waves traveling on the string would be reflected at both ends; the vibration of the string would then be the combination of such waves moving to and fro along the string and *standing waves* would be formed.

3 Terms related to wave motion

Wave length λ

It is defined as the distance travelled by a wave during the time particle executing simple harmonic motion (SHM) completes one vibration. It is the distance between two consecutive particles executing SHM in same phase. It is the distance between two consecutive crests or troughs.

VELOCITIES IN WAVE MOTION

There are three velocities in wave motion which are quite distinct although they are connected mathematically. They are

1. *The particle velocity*, which is the simple harmonic velocity of the oscillator about its equilibrium position.

2. *The wave or phase velocity*, the velocity with which planes of equal phase, crests or troughs, progress through the medium.

3. *The group velocity.* A number of waves of different frequencies, wavelengths and velocities may be superposed to form a group. Waves rarely occur as single monochromatic components; a white light pulse

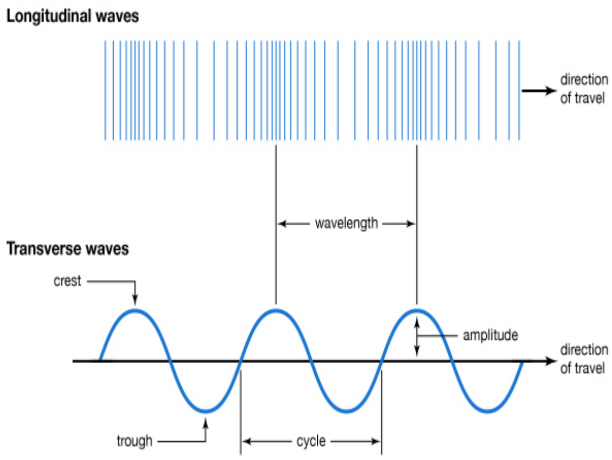

Figure 4: Various terms depicted in Longitudinal and transverse wave motion

consists of an infinitely fine spectrum of frequencies and the motion of such a pulse would be described by its group velocity. Such a group would, of course, 'disperse' with time because the wave velocity of each component would be different in all media except free space. Only in free space would it remain as white light. Its importance is that it is the velocity with which the energy in the wave group is transmitted. For a monochromatic wave the group velocity and the wave velocity are identical. Here we shall concentrate on particle and wave velocities.

4 Traveling waves

A common experience is to take the end of a long rope like a clothesline and move one end of it up and down rapidly to launch a wave pulse down the rope. A schematic diagram of this is shown in Figure 5. The pulse roughly holds its shape and travels with a definite velocity along the rope. Here we will use a Gaussian function to model this travelling wave pulse. The Gaussian function can be represented by

$$y = A \exp -(x^2/a^2) \tag{1}$$

where A and a are constants. This function appears in many branches of the physical sciences and is plotted in Figure6. When $x = 0, y = A$ and when $x = \pm a, y = A/e$. A corresponds to the height of the Gaussian and a is a measure of its width. If we now change the variable x to $(x - b)$ we obtain

$$y = A \exp -(x - b)^2/a^2 \tag{2}$$

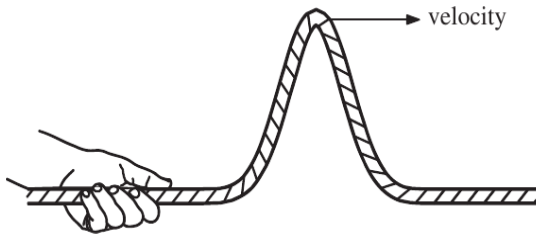

Figure 5: A wave pulse can be launched down a long rope by moving the end of the rope rapidly up and down

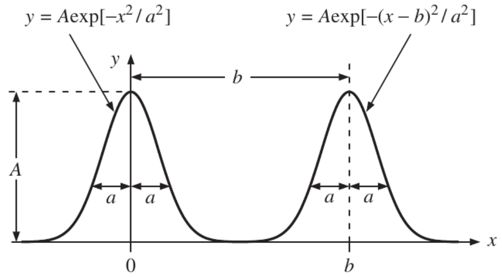

Figure 6: The Gaussian functions $y = A\exp-(x^2/a^2)$ and $y = A\exp-(x-b)^2/a^2$. A is the height of the Gaussian and a characterises its width. These two Gaussians have the same shape but are separated by distance b.

This function is also plotted in Figure 6. We see that the shape of the function, as characterised by its height and width, is the same as before. We have simply moved the Gaussian a distance b to the left, so that now it has its maximum value A at $x = b$. Suppose we now change the variable x to $(x - vt)$ where t is time and v is a constant with the dimensions of distance/time. Then we obtain

$$y(x,t) = A\exp-(x-vt)^2/a^2 \tag{3}$$

The value of vt increases linearly with time. Consequently, Equation (3) describes a Gaussian that moves in the positive x-direction at a constant rate just like the wave pulse on the rope. This is illustrated in Figure 7 where the Gaussian is plotted at three different instants of time that are separated by equal time intervals of δt. The rate at which it moves is the velocity v.

We can generalise the above by saying that when a wave is going in the positive x-direction, the dependence of the shape of the rope on x and t must be of the general form $f(x - vt)$, where f is some function of $(x - vt)$.

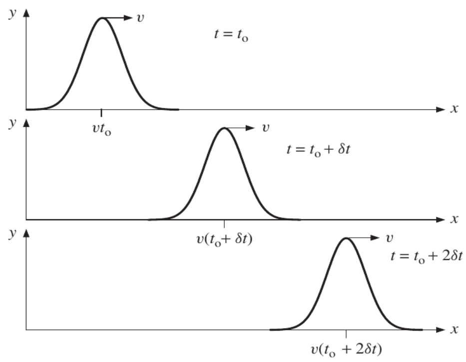

Figure 7: The Gaussian $y = A \exp -(x - vt)^2/a^2$ plotted as a function of position x, at three different instants of time, separated by equal time intervals of δt.

Examples of $f(x - vt)$ are the Gaussian function $A \exp -(x - vt)^2/a^2$ that we saw above, and the travelling sinusoidal wave $A \sin 2\pi(x - vt)/\lambda$ that we will discuss in the next section. The shape of the wave is given by $f(x - vt)$ at $t = 0$, i.e. by $f(x)$ as illustrated in Figure 8(a). At time t, the wave has moved a distance vt to the right. However it has retained its shape, as shown in Figure 8(b). This is the important characteristic of wave motion: the wave retains its shape as it travels along. Clearly, we could determine the shape of the wave by taking a snapshot of the rope at a particular instant of time. However, we could also find this shape by measuring the variation in the displacement of a given point on the rope as the wave passes by. A wave travelling in the negative x-direction must be of the general form $g(x + vt)$ where g is some function of $(x + vt)$. Again at $t = 0$, $g(x)$ gives the shape of the wave as illustrated in Figure 9(a). At time t, the wave has moved to the left by a distance vt but its shape remains the same, as shown in Figure 9(b). The general form of any wave motion of the rope can be written as

$$y = f(x - vt) + g(x + vt) \tag{4}$$

and can be considered as a superposition of two waves, each of speed v, travelling in opposite directions.

4.1 Travelling sinusoidal waves

Sinusoidal waves are important because they occur in many physical situations, such as in the propagation of electromagnetic radiation. They are also important because more complicated wave shapes can be decomposed into a combination of sinusoidal waves. Consequently, if we understand sinusoidal waves we can understand these more complicated waves. A travelling sinusoidal wave is illustrated in Figure 10, at various instants of time. The

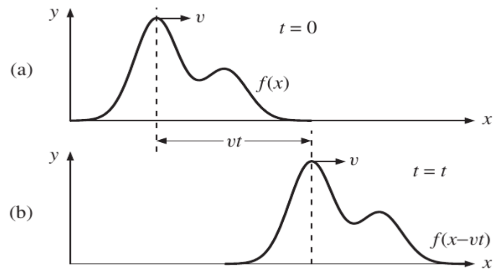

Figure 8: A wave travelling in the positive x-direction, defined by the function $y = f(x - vt)$. (a)$f(x - vt) \equiv f(x)$ at time $t = 0$, which gives the shape of the wave. (b) $f(x - vt)$ at time t when the wave has moved a distance vt to the right.

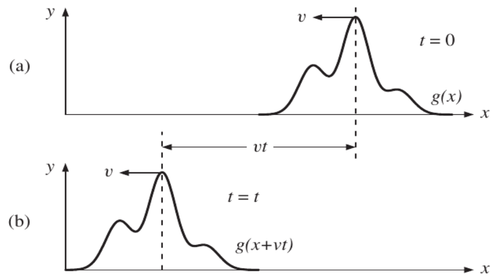

Figure 9: A wave travelling in the negative x-direction, defined by the function $y = g(x + vt)$. (a)$g(x + vt) \equiv g(x)$ at time $t = 0$. (b) $g(x + vt)$ at time t when the wave has moved a distance vt to the left.

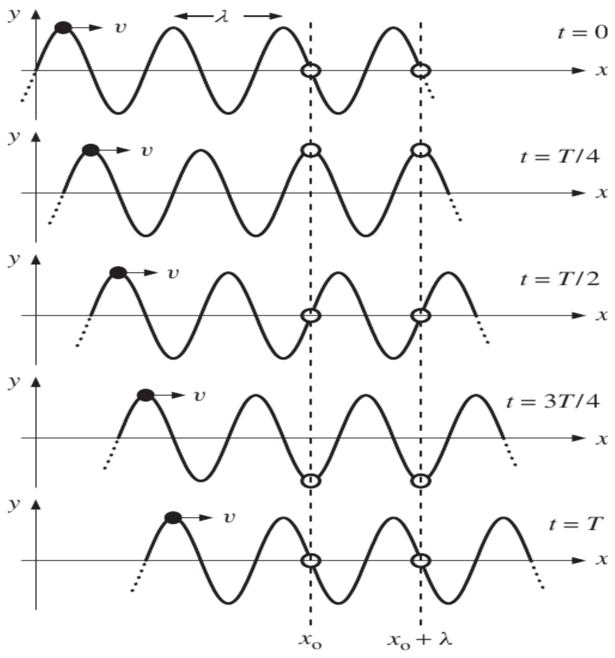

Figure 10: Schematic representation of a travelling sinusoidal wave of wavelength λ and period T , at the different times as indicated. Each point on the wave travels at velocity v. The open circles denote points on the wave that are separated by wavelength λ. These points move in phase with each other in the transverse direction.

dotted parts of the curves indicate that the wave extends a large distance in both directions to avoid any effects due to reflections of the wave at a fixed end. A sinusoidal wave is a repeating pattern. The length of one complete pattern is the distance between two successive maxima (crests), or between any two corresponding points. This repeat distance is the wavelength λ of the wave. The sinusoidal wave propagates along the x-direction and the displacement is in the y-direction, at right angles to the propagation direction. We could generate such a sinusoidal wave by moving the end of a long rope up and down in simple harmonic motion. The displacements lie in a single plane, i.e. in the x-y plane, and so we describe the waves as linearly polarised in that plane. We represent the travelling sinusoidal wave by

$$y(x,t) = A \sin \frac{2\pi}{\lambda}(x - vt) \tag{5}$$

where A is the amplitude and λ is the wavelength. This function repeats itself each time x increases by the distance λ. At $t = 0$, we have $y = A\sin(2\pi x/\lambda)$ which shows the sinusoidal shape of the wave. The transverse displacement y given by Equation (5) is a function of two variables x and t and it is interesting to see what happens if we keep either x or t fixed. Keeping x fixed is like watching a leaf on a pond that bobs up and down with the motion of the water ripples. Keeping t fixed is like taking a snapshot of the pond that fixes the positions of the water ripples in time. The sinusoidal wave travels at a definite velocity v in the positive x-direction, as can be seen from the progression of a wave crest with time in Figure 11. The number of times per unit time that a wave crest passes a fixed point, at say $x = x_o$, is the *frequency* ν of the wave. The frequency ν is equal to the velocity v of the wave divided by the wavelength λ. Hence we obtain

$$\nu\lambda = v \tag{6}$$

We see that the important parameters of the wave (wavelength, frequency and velocity) are related by this simple equation. The time T that a wave crest takes to travel a distance λ is equal to λ/v, i.e. the reciprocal of the frequency. Hence,

$$\nu = \frac{1}{T} \tag{7}$$

where T is the *time period* of the wave.

Figure 11 also illustrates how the displacement of a point on the wave, at $x = x_o$, changes with time. The point moves up and down as the wave passes by and indeed its motion is simple harmonic. We can see this mathematically as follows. We have

$$y(x,t) = A \sin \frac{2\pi}{\lambda}(x - vt) \tag{8}$$

Then at the fixed position, $x = x_o$, we have

$$y(x_o,t) = A \sin \frac{2\pi}{\lambda}(x_o - vt) \tag{9}$$

Now since x has a fixed value and we want to see how y varies with t it is useful to write this equation in the equivalent form

$$y(x_o,t) = -A \sin \frac{2\pi}{\lambda}(vt - x_o) \tag{10}$$

using the relationship $\sin(\alpha - \beta) = -\sin(\beta - \alpha)$. Equation (10) shows that the displacement varies sinusoidally with time t with an angular frequency ω where

$$\omega = \frac{2\pi v}{\lambda} = 2\pi\nu. \tag{11}$$

Each point on the wave completes one period of oscillation in time period T , and we emphasise that all points along the wave oscillate at the same frequency ω. We can consider the term $2\pi x_o/\lambda$ in Equation (10) as a phase angle. Thus, as illustrated in Figure 11, points at $x = x_o$ and $x = x_o + \lambda$, denoted by the open circles, oscillate in phase with each other. As the wave propagates, any particular point on it, for example the wave crest denoted by the bold dots in Figure 11, maintains a constant value of transverse displacement y, and hence a constant value of $(x - vt)$. Since $(x - vt) =$ constant, $dx/dt = v$, which of course is the wave velocity.

We can use Equation (8) to obtain alternative mathematical expressions for the wave. Substituting for $v = \nu\lambda$ in Equation (8) we obtain

$$y(x,t) = A\sin\left(\frac{2\pi x}{\lambda} - 2\pi\nu t\right) \tag{12}$$

We define the quantity $2\pi/\lambda$ as the *wavenumber* k, i.e.

$$k = 2\pi/\lambda. \tag{13}$$

Substituting for $\omega = 2\pi\nu$ from Equation (11) and k from Equation (13) in Equation (12), we obtain

$$y(x,t) = Asin(kx - \omega t). \tag{14}$$

In addition, using the relationships $\nu\lambda = v$ and $2\pi\nu = \omega$, we have

$$v = \frac{\omega}{k} \tag{15}$$

The wave velocity is equal to the angular frequency divided by the wavenumber. Although we have used sine functions, we can equally well use cosine functions such as

$$y(x,t) = A\cos(kx - \omega t), \tag{16}$$

since the cosine function is simply the sine function with a phase difference of $\pi/2$. This is illustrated in Figure 11, which shows snapshots of Equations (14) and (16) at $t = 0$. We simply need to choose the solution that fits the initial conditions. Finally, as we know that it can be advantageous to use a complex representation of periodic motion. This is also the case for wave motion, remembering that, as usual, the real part of the complex form is the physical quantity. Thus we can write the following alternative mathematical expressions for travelling sinusoidal waves:

$$y(x,t) = A\exp\frac{2\pi}{\lambda}i(x - vt). \tag{17}$$

$$y(x,t) = A\exp 2\pi i(\frac{x}{\lambda} - \nu t). \tag{18}$$

$$y(x,t) = A\exp i(kx - \omega t). \tag{19}$$

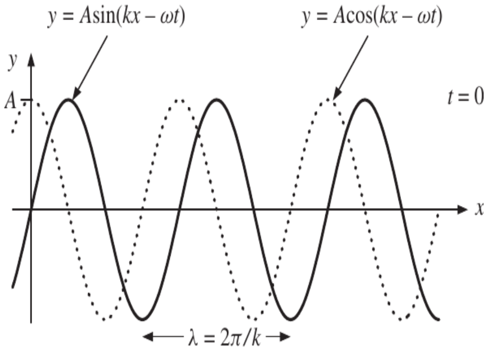

Figure 11: Representation of the functions $y = A\sin(kx - \omega t)$ and $y = A\cos(kx - \omega t)$ at time $t = 0$, showing the phase relationship between the two functions.

4.2 THE WAVE EQUATION

In equation 4 we saw that the general form of any wave motion is given by

$$y = f(x - vt) + g(x + vt).$$

We now show that this is the general solution of the wave equation. We start with the function $f(x - vt)$ and change variables to $u = (x - vt)$ to obtain the function $f(u)$. Notice that $f(u)$ is a function only of u. Then

$$\frac{\partial f}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x}$$

and

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{df}{du}\frac{\partial u}{\partial x}\right) = \frac{d^2 f}{du^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{df}{du}\frac{\partial^2 u}{\partial x^2}$$

Since $u = (x - vt) \Rightarrow \partial u/\partial x = 1$ and $\partial^2 u/\partial x^2 = 0$, we have

$$\frac{\partial^2 f}{\partial x^2} = \frac{d^2 f}{du^2}. \tag{20}$$

Similarly,

$$\frac{\partial f}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t}$$

and

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{df}{du}\frac{\partial u}{\partial t}\right) = \frac{d^2 f}{du^2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{df}{du}\frac{\partial^2 u}{\partial t^2}$$

Since $u = (x - vt) \Rightarrow \partial u/\partial t = -v$ and $\partial^2 u/\partial t^2 = 0$, we have

$$\frac{\partial^2 f}{\partial t^2} = v^2\frac{d^2 f}{du^2}. \tag{21}$$

Combining Equations (20) and (21) we obtain

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$

Similarly, we can readily see that $g(x + vt)$ satisfies the equation

$$\frac{\partial^2 g}{\partial t^2} = v^2 \frac{\partial^2 g}{\partial x^2} \tag{22}$$

[It does not matter that the sign of the velocity has changed between $f(x-vt)$ and $g(x+vt)$ since only the square of the velocity occurs in Equation (22).] Thus

$$\frac{\partial^2 (f+g)}{\partial t^2} = v^2 \frac{\partial^2 (f+g)}{\partial x^2}$$

and hence we can write

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \qquad (23)}$$

This is a fundamental result. Equation (23) is the *one-dimensional wave equation.* (The position of the velocity v in Equation (5.23) is consistent with the dimensions of the quantities involved.) The general solution of it is Equation (4), namely

$$y = f(x - vt) + g(x + vt).$$

The wave equation (23) and its general solution apply to all waves that travel in one dimension. For example, they describe sound waves in a long tube where the relevant physical parameter is the local air pressure $P(x,t)$. They describe voltage waves $V(x,t)$ on a transmission line and temperature fluctuations $T(x,t)$ along a metal rod. Consequently we write the wave equation more generally as

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

and its general solution as

$$\psi = f(x - vt) + g(x + vt).$$

where ψ represents the relevant physical quantity.

As a specific example of the above discussion, we have the travelling sinusoidal wave $y = A \sin [2\pi(x - vt)/\lambda]$. First, differentiating with respect to x and keeping t constant, we obtain

$$\frac{\partial y}{\partial x} = \left(\frac{2\pi}{\lambda}\right) A \cos [2\pi(x - vt)/\lambda]$$

and

$$\frac{\partial^2 y}{\partial x^2} = -\left(\frac{2\pi}{\lambda}\right)^2 A^2 \sin [2\pi(x - vt)/\lambda] \tag{24}$$

Similarly,

$$\frac{\partial y}{\partial t} = -v \left(\frac{2\pi}{\lambda}\right) A \cos [2\pi(x - vt)/\lambda]$$

and

$$\frac{\partial^2 y}{\partial t^2} = -v^2 \left(\frac{2\pi}{\lambda}\right)^2 A^2 \sin [2\pi(x - vt)/\lambda] \tag{25}$$

Finally, dividing Equation (25) by Equation (24) we obtain the expected result,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$