Lecture notes of Physics Hons. Paper V, Group A

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1 Scalar and Vector field

1.1 Scalar field

If to each point (x, y, z) a region R in space there corresponds a number or a scalar $\Phi(x, y, z)$ then Φ is called a scalar field function of position or scalar field and we say that a scalar field Φ has been defined in the region R.
Examples:
1. Temperature at any point on a surface at a certain time defines a scalar field.
2. $\Phi(x, y, z) = x^3y - z^2$ defines a scalar field.
A scalar field which is independent of time is called stationary or steady state scalar field.

1.2 Vector field

If to each point (x, y, z) a region R in space there corresponds a number or a vector $\vec{V}(x, y, z)$ then $\vec{V}$ is called a vector field function of position or vector point function and we say that a vector field $\vec{V}$ has been defined in the region R.
Examples:
1.If the velocity at any point (x, y, z) within a moving fluid is known at a certain time, then a vector field is defined.
2. $\vec{V}(x, y, z) = xy^2\hat{i} - 2yz^3\hat{j} + x^2z\hat{k}$ defines a vector field.
A vector field which is independent of time is called stationary or steady state vector field.

Gradient, Divergence and Curl of Fields

Let us first define a vector differential operator $\vec{\nabla}$ (pronounced as dell). $\vec{\nabla}$ is defined as

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \tag{1}$$

This vector operator possesses property analogous to to those of all ordinary vectors. It is useful in defining three quantities which arise in practical applications and are known as the gradient ($\vec{\nabla}$), divergence ($\vec{\nabla}.$) and curl ($\vec{\nabla}x$).

2 The Gradient

Let $\Phi(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space i.e. Φ defines a differentiable scalar field. Then the gradient of Φ, written as grad Φ or $\vec{\nabla}\Phi$, is defined as

$$\vec{\nabla}\Phi = \frac{\partial \Phi}{\partial x}\hat{i} + \frac{\partial \Phi}{\partial y}\hat{j} + \frac{\partial \Phi}{\partial z}\hat{k} \tag{2}$$

Example:
If $\Phi(x, y, z) = 3x^2y - y^3z^2$, find grad Φ at the point $(1, -2, 1)$.
Sol.:

$$\frac{\partial\Phi}{\partial x} = 6xy$$
$$\frac{\partial\Phi}{\partial y} = 3x^2 - 3y^2z^2$$
$$\frac{\partial\Phi}{\partial z} = -2zy^3$$

$\Rightarrow \vec{\nabla}\Phi = 6xy\hat{i} + 3(x^2 - y^2z^2)\hat{j} - 2zy^3\hat{k}$.

Let us substitute $(x, y, z) = (1, -2, 1)$ in the above equation to find grad Φ at the point $(1, -2, 1)$
$\vec{\nabla}\Phi = -12\hat{i} - 3\hat{j} + 16\hat{k}$. *(Answer)*

Note: While Φ is a scalar field, Grad Φ defines a vector field.

Home work:
1. Find the gradient of $\Phi = \log r$ where r is magnitude of position vector.
2. Find the gradient of $\Phi = \frac{1}{r}$ where r is magnitude of position vector.

2.1 Geometrical Meaning of Gradient

Let us consider a scalar function $\Phi(x, y, z)$. If we move from (x, y, z) to $(x = dx, y + dy, z + dz)$, the scalar function Φ changes by an amount

$$d\Phi = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz \tag{3}$$

Above equation is a standard theorem of partial differentiation. R.H.S. of above equation can also be written as $\left(\frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k}\right).(dx\hat{i} + dy\hat{j} + dz\hat{k}) = \vec{\nabla}\Phi.d\vec{r}$.
Thus we obtain

$$d\Phi = \vec{\nabla}\Phi.d\vec{r} \tag{4}$$

If we choose $\Phi(x, y, z)$=constant surface,

$$d\Phi = 0$$

So, from Eqn. (4), we get

$$\vec{\nabla}\Phi.d\vec{r} = 0 \tag{5}$$

Above equation shows that $\vec{\nabla}\Phi$ is perpendicular to $d\vec{r}$ vector which is tangent vector.
$\Rightarrow \vec{\nabla}\Phi$ is perpendicular to the tangent vector. As we know that a line perpendicular to tangent at a point on a surface is known as normal. Therefore $\vec{\nabla}\Phi$ or gradient of Φ represents the normal to the $\Phi(x, y, z)$ surface.

3 Divergence

Let $\vec{V}(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space i.e. $\vec{V}$ defines a differentiable scalar field. Then the divergence of $\vec{V}$, written as $\vec{\nabla}.\vec{V}$, is defined as

$$\vec{\nabla}.\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \tag{6}$$

Note the analogy with $\vec{A}.\vec{B} = A_1B_1 + A_2B_2 + A_3B_3$. The difference is that

$$\vec{A}.\vec{B} = \vec{B}.\vec{A} \text{ but } \vec{\nabla}.\vec{V} \neq \vec{V}.\vec{\nabla}$$

Divergence of a vector field is a scalar field.
Any vector for which

$$\vec{\nabla}.\vec{A} = 0$$

$\vec{A}$ is called Solenoidal vector.
Examples:
1. If $\vec{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, find $\vec{\nabla}.\vec{A}$ at point $(1,-1,1)$.
Sol:

$$\because \vec{\nabla}.\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$\Rightarrow \vec{\nabla}.\vec{A} = 2xz - 6y^2z^2 + xy^2$$

At point $(x,y,z) = (1,-1,1)$

$$\vec{\nabla}.\vec{A} = 2 - 6 + 1 = -3$$

2.Show that $\vec{\nabla}.(\vec{\nabla}\Phi) = \vec{\nabla}^2\Phi$ where

$$\vec{\nabla}^2\Phi = \tfrac{\partial^2\Phi}{\partial x^2} + \tfrac{\partial^2\Phi}{\partial y^2} + \tfrac{\partial^2\Phi}{\partial z^2}$$

$\vec{\nabla}^2$ is known as Laplacian operator.
Proof:
By definition of gradient

$$\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k}$$

Let $\vec{\nabla}\Phi$ be called a new vector $\vec{V}$.

$$\Rightarrow \vec{\nabla}.(\vec{\nabla}\Phi) = \vec{\nabla}.\vec{V}$$

From the definition of divergence

$$\vec{\nabla}.\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Substituting V_1, V_2 and V_3 from the equation $\vec{V} = \vec{\nabla}\Phi$

$$\vec{\nabla}.\vec{V} = \frac{\partial\frac{\partial\Phi}{\partial x}}{\partial x} + \frac{\partial\frac{\partial\Phi}{\partial y}}{\partial y} + \frac{\partial\frac{\partial\Phi}{\partial z}}{\partial z}$$

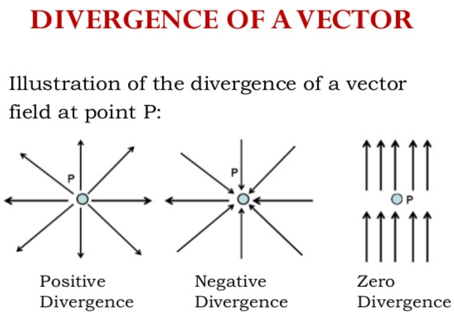
Or

$$\vec{\nabla}.\vec{V} = \vec{\nabla}.(\vec{\nabla}\Phi) = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

The R.H.S. of the above equation is nothing but $\vec{\nabla}^2\Phi$ from the definition of the Laplacian operator. Thus we have proved

$$\vec{\nabla}.(\vec{\nabla}\Phi) = \vec{\nabla}^2\Phi \tag{7}$$

Homework:
1. If $\Phi = 2x^3y^2z^4$, find $\vec{\nabla}.(\vec{\nabla}\Phi)$ and show that it is equal to $\vec{\nabla}^2\Phi$.
2. Find the divergence of position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.



3.1 Physical interpretation of divergence

The divergence can be considered as a quantitative measure of how much a vector field diverges (spreads out) or converges at any give point.

For example, if we consider velocity field $\vec{v}(x, y, z)$ describing the local velocity at any point in a fluid, then the divergence of $\vec{v}$ is equal to the net rate of outflow of fluid per unit volume evaluated at a point.

4 Curl

If $\vec{A}(x, y, z)$ is a differentiable vector field, then the curl of $\vec{A}$ is defined as following:

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

where $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$. Or,

$$\nabla \times \vec{A} = \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \hat{j}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \tag{8}$$

Example:
1. Find the curl of $\vec{A}$ where $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$ at point $(1, -1, 1)$.
Sol:

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$\Rightarrow \nabla \times \vec{A} = (2z^4 - 2x^2y)\hat{i} + (3z^2x + 4yz^2)\hat{j} + 4xyz\hat{k}$$

At point $(1, -1, 1)$

$$\nabla \times \vec{A} = 3\hat{j} + 4\hat{k}.$$

2. Find the curl of position vector $\vec{r}$.
Sol: The position vector $\vec{r}$ is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow \vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

3. **Prove that curl of gradient of any scalar field Φ is always zero** i.e. $\vec{\nabla} \times \vec{\nabla}\Phi = 0$.
Proof: Let Grad $\Phi = \vec{\nabla}\Phi$ be called a new vector $\vec{A}$. So, from the definition of gradient

$$\vec{A} = \vec{\nabla}\Phi = \frac{\partial \Phi}{\partial x}\hat{i} + \frac{\partial \Phi}{\partial y}\hat{j} + \frac{\partial \Phi}{\partial z}\hat{k}$$

$$\Rightarrow \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} & \frac{\partial \Phi}{\partial z} \end{vmatrix}$$

Or,

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{\nabla}\Phi = \left(\frac{\partial^2 \Phi}{\partial z \partial y} - \frac{\partial^2 \Phi}{\partial y \partial z}\right) + \hat{j}\left(\frac{\partial^2 \Phi}{\partial x \partial z} - \frac{\partial^2 \Phi}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 \Phi}{\partial y \partial x} - \frac{\partial^2 \Phi}{\partial x \partial y}\right)$$

Since the order of partial derivatives does not matter, i.e. $\frac{\partial^2 \Phi}{\partial x \partial y} = \frac{\partial^2 \Phi}{\partial y \partial x}$ and so on, from the above equation we get

$$\vec{\nabla} \times \vec{\nabla}\Phi = 0.(Proved)$$

Thus we prove that curl of gradient of any scalar field is always zero.

4. **Prove that divergence of curl of any vector is always zero.**
Proof: Let $\vec{A}$ be a vector field. Then

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

where $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$. Or,

$$\nabla \times \vec{A} = \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \hat{j}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

Let use call $\nabla \times \vec{A}$ a new vector $\vec{V}$ i.e. $\vec{V} = \nabla \times \vec{A}$. Our aim is to find the divergence of curl of $\vec{A}$ i.e. $\nabla.(\nabla \times \vec{A}) = \nabla.\vec{V}$.
By definition, divergence of $\vec{V}$ is

$$\vec{\nabla}.\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Substituting V_1, V_2 and V_3 from the expression of $\vec{V} = \nabla \times \vec{A}$, we obtain

$$\vec{\nabla}.\vec{V} = \frac{\partial\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)}{\partial x} + \frac{\partial\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)}{\partial y} + \frac{\partial\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)}{\partial z}$$

Or,

$$\vec{\nabla}.\vec{V} = \left(\frac{\partial^2 A_3}{\partial x\partial y} - \frac{\partial^2 A_2}{\partial x\partial z}\right) + \left(\frac{\partial^2 A_1}{\partial y\partial z} - \frac{\partial^2 A_3}{\partial y\partial x}\right) + \left(\frac{\partial^2 A_2}{\partial z\partial x} - \frac{\partial^2 A_1}{\partial z\partial y}\right) = 0.$$

Thus we prove that the divergence of a curl is always zero.

5. **If $\vec{v} = \vec{\omega} \times \vec{r}$, prove that $\omega = \frac{1}{2}\nabla \times \vec{v}$ where ω is a constant vector.**
Proof:
The position vector $\vec{r}$ is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

and $\omega = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$

$$\Rightarrow \vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = (\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}$$

$$\Rightarrow \vec{\nabla}\times\vec{v} = \vec{\nabla}\times(\vec{\omega}\times\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix} = (\omega_1+\omega_1)\hat{i}+(\omega_2+\omega_2)\hat{j}+(\omega_3+\omega_3)\hat{k}$$

$$\Rightarrow \vec{\nabla} \times \vec{v} = \vec{\nabla} \times (\vec{\omega} \times \vec{r}) = (\omega_1 + \omega_1)\hat{i} + (\omega_2 + \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k} = 2\vec{\omega}$$

Or,

$$\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{v} = \frac{1}{2}\vec{\nabla} \times (\vec{\omega} \times \vec{r}) \quad \text{Proved.} \tag{9}$$

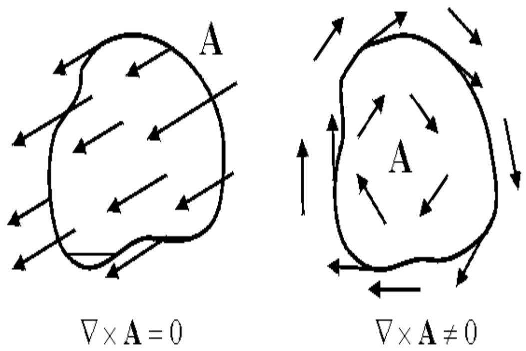


Figure 1: Examples of irrotational and vortex vector fields

4.1 Physical interpretation of curl

For a vector field $\vec{v}(x, y, z)$ describing the local velocity at any point in a fluid, curl of $\vec{v}$ is a measure of the angular velocity of the fluid in the neighbourhood of that point.If a paddle wheel were placed at various points in the fluid then it would tend to rotate in regions where $\nabla \times \vec{v}$ is non zero i.e. $(\nabla \times \vec{v} \neq 0)$ while it would not rotate in regions where curl of $\vec{v}$ is equal to zero i.e. $(\nabla \times \vec{v} = 0)$.

If $(\nabla \times \vec{V} = 0)$, then $\vec{V}$ is called irrotational vector and if $(\nabla \times \vec{V} \neq 0)$ then $\vec{V}$ is called vortex field.

5 Some important formulae involving differential operator $\vec{\nabla}$

If $\vec{A}$ and $\vec{B}$ are two differentiable vector functions and, ϕ and ψ are two differentiable scalar functions of (x, y, z), then

1. $\vec{\nabla}(\phi + \psi) = \vec{\nabla}\phi + \vec{\nabla}\psi$
2. $\vec{\nabla}.(\vec{A} + \vec{B}) = \vec{\nabla}.\vec{A} + \vec{\nabla}.\vec{B}$
3. $\vec{\nabla} \times (\vec{A} + \vec{B}) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{B}$
4. $\vec{\nabla}.(\phi\vec{A}) = (\vec{\nabla}\phi).\vec{A} + \phi(\vec{\nabla}.\vec{A})$
5. $\vec{\nabla} \times (\phi\vec{A}) = (\vec{\nabla}\phi) \times \vec{A} + \phi(\vec{\nabla} \times \vec{A})$
6. $\vec{\nabla}.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{\nabla} \times \vec{A}) - \vec{A}.(\vec{\nabla} \times \vec{B})$
7. $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} - (\vec{A}.\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}.\vec{B}) - \vec{B}(\vec{\nabla}.\vec{A})$
8. $\vec{\nabla}(\vec{A}.\vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} + (\vec{A}.\vec{\nabla})\vec{B} + \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A})$
9. $\vec{\nabla}.\vec{\nabla}\phi = \vec{\nabla}^2\phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$; $\vec{\nabla}^2$ is known as Laplacian operator.
10. $\vec{\nabla} \times \vec{\nabla}\phi = 0$
11. $\vec{\nabla}.(\vec{\nabla} \times \vec{A}) = 0$
12. $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - (\vec{\nabla}.\vec{\nabla})\vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}^2\vec{A}$

6 Gradient, Divergence and Curl in spherical polar coordinates

Let us suppose a vector $\vec{A}$ whose components in Cartesian and Spherical polar coordinates are as following:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi} \tag{10}$$

Divergence operator in Cartesian coordinates is defined as:

$$\vec{\nabla}.\vec{A} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right).\vec{A} \tag{11}$$

The cartesian coordinates (x, y, z)can be obtained from the Spherical Polar coordinates (r, θ, ϕ) using the following transformation relations:

$$x = r\sin\theta\cos\phi \tag{12}$$

$$y = r\sin\theta\sin\phi \tag{13}$$

$$z = r\cos\theta \tag{14}$$

And the Spherical Polar coordinates (r, θ, ϕ) can be obtained from the cartesian coordinates (x, y, z) using the following transformation relations:

$$r = \sqrt{x^2 + y^2 + z^2} \tag{15}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \tag{16}$$

$$\phi = \tan^{-1}\frac{y}{x} \tag{17}$$

Since the partial derivates $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ appear in the definition of divergence (equation 11), we find how these partial derivatives transform in spherical polar coordiantes using the chain rule of differentiation,

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \tag{18}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \tag{19}$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} \tag{20}$$

From equations (15)-(17) we obtain,

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta\cos\phi \tag{21}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \tag{22}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} \tag{23}$$

$$\frac{\partial \theta}{\partial x} = \frac{xz}{r^2\sqrt{x^2 + y^2}} \tag{24}$$

$$\frac{\partial \theta}{\partial y} = \frac{yz}{r^2\sqrt{x^2 + y^2}} \tag{25}$$

$$\frac{\partial \theta}{\partial z} = -\frac{\sqrt{x^2 + y^2}}{r^2} \tag{26}$$

$$\frac{\partial \phi}{\partial x} = -\frac{y}{x^2 + y^2} \tag{27}$$

$$\frac{\partial \phi}{\partial y} = \frac{x}{x^2 + y^2} \tag{28}$$

$$\frac{\partial \phi}{\partial z} = 0 \tag{29}$$

Using equations(12)-(14) into above equations we obtain

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi \tag{30}$$

$$\frac{\partial r}{\partial y} = \sin\theta\sin\phi \tag{31}$$

$$\frac{\partial r}{\partial z} = \cos\theta \tag{32}$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\theta\cos\phi}{r} \tag{33}$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta\sin\phi}{r} \tag{34}$$

$$\frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r} \tag{35}$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta} \tag{36}$$

$$\frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\theta} \tag{37}$$

$$\frac{\partial \phi}{\partial z} = 0 \tag{38}$$

Substituting above equations in equations (18)-(20) we obtain

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{39}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{40}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \tag{41}$$

In equation (11) we have transformed the partial derivatives from cartesian to spherical polar coordinates. Now we have to transform the unit vectors. Since we know that the unit vectors $(\hat{e_1}, \hat{e_2}, \hat{e_3})$ in an orthogonal curvilinear corrdinates (u_1, u_2, u_3) are defined as

$$\hat{e_i} = \frac{\frac{\partial \vec{r}}{\partial u_i}}{|\frac{\partial \vec{r}}{\partial u_i}|} \tag{42}$$

where $i = 1, 2, 3$ and $\vec{r}$ is the position vector.

$$\Rightarrow \hat{r} = \frac{\frac{\partial \vec{r}}{\partial r}}{|\frac{\partial \vec{r}}{\partial r}|} \tag{43}$$

$$\hat{\theta} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{|\frac{\partial \vec{r}}{\partial \theta}|} \tag{44}$$

$$\hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{|\frac{\partial \vec{r}}{\partial \phi}|} \tag{45}$$

Since $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z}$

$$\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{46}$$

$$\Rightarrow |\frac{\partial \vec{r}}{\partial r}| = 1 \tag{47}$$

$$\Rightarrow \hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{48}$$

Similarly,

$$\frac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\phi\hat{x} + r\cos\theta\sin\phi\hat{y} - r\sin\theta\hat{z} \tag{49}$$

$$\Rightarrow |\frac{\partial \vec{r}}{\partial \theta}| = r \tag{50}$$

$$\Rightarrow \hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \tag{51}$$

and

$$\frac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\hat{x} + r\sin\theta\cos\phi\hat{y} \tag{52}$$

$$\Rightarrow |\frac{\partial \vec{r}}{\partial \phi}| = r\sin\theta \tag{53}$$

$$\Rightarrow \hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \tag{54}$$

By doing some simple algebra we can find $(\hat{x}, \hat{y}, \hat{z})$ in terms of $(\hat{r}, \hat{\theta}, \hat{\phi})$ as following:

$$\hat{x} = \sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi} \tag{55}$$

$$\hat{y} = \sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi} \tag{56}$$

$$\hat{z} = \cos\theta\hat{r} + \sin\theta\hat{\theta} \tag{57}$$

Substituting equations (39)-(41) and (55)-(57) into equation (11), we obtain

$$\begin{aligned}\vec{\nabla}.\vec{A} = \quad & \Bigg[\left(\sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi}\right)\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \\ & + \left(\sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi}\right)\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \\ & + \left(\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\Bigg].\vec{A}\end{aligned} \tag{58}$$

Upon simplifying above equation, we see that most terms cancel out and we get

$$\vec{\nabla}.\vec{A} = \left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right].\vec{A} \tag{59}$$

$$\Rightarrow \vec{\nabla} = \left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right] \tag{60}$$

Eqn (59) represents the divergence of a vector $\vec{A}$ in spherical polar coordinates. We derived it starting from the expression of $\vec{\nabla}.\vec{A}$ in cartesian coordiantes. And Eqn (60) represents the differential operator $\vec{\nabla}$ in spherical polar coordinates.

Using Eqn (60) we can obtain the gradient of a scalar field and curl of a vector field in spherical polar cooordinates.

Let us suppose a scalar field Φ and a vector field $\vec{A}$. From the definition of gradient and its expression in spherical polar coordinates in eqn (60) the gradient of scalar field Φ is obtained to be

$$\vec{\nabla}\Phi = \left[\hat{r}\frac{\partial\Phi}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right] \tag{61}$$

Similarly the curl of a vector field $\vec{A}$ is found to be

$$\vec{\nabla}\times\vec{A} = \left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\times\vec{A} \tag{62}$$

Or, curl of $\vec{A}$ in the determinant form is as following:

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\ A_r & A_\theta & A_\phi \end{vmatrix}$$

Homework: Using the method in section 6 derive the expressions for gradient, divergence and curl in cylindrical coordinates.