Topic 3: Phase Space, Costraints & Hamilton's Equation, B.Sc. (Hons.), Part -1, Paper 1

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Phase space,

It is defined as a space of $2s$ dimensions whose axes of coordinates are the s generalized coordinates and the s momenta of the given system. Each point of this space corresponds to a definite mechanical state of the system. when the system is in motion, the representative point in phase space performs a curve called *phase trajectory.*

- Space of configurations

The state of a system composed of n particles under the action of m constraints connecting some of the $3n$ cartesian coordinates is completely determined by $s = 3n - m$ generalized coordinates. Thus, it is possible to descibe the state of such a system by a point in the s dimensional space usually called the *configuration space*, for which each of its dimensions corresponds to one q_j. The time evolution of the system will be represented by a curve in the configuration space made of points describing the instantaneous configuration of the system.

- Constraints

Let us consider the *constraints* that act on the motion of the system. The constraints can be classified in number of ways. In the general constraint equations :

$$\sum_i c_{\alpha i} \dot{q_i} = 0 \ ,$$

where the $c_{\alpha i}$ are functions of only the coordinates (the index α is the number of constraint equations). If the first terms of such equations are not total derivatives with respect to the time they cannot be integrated. Hence we may say that they cannot be reduced to relationships between only the coordinates, that might be used to express the position by less coordinates, corresponding to the real number of degrees of freedom. Such constraints are called *non holonomic* and other which are *holonomic*, which connects only the coordinates of the system.

. Hamilton's equations of motion

As we know from the formulation of the laws of Mechanics we assume in Lagrangian that the mechanical state of the system is determined by its generalized coordinates and velocities.

From one set of independent variables to another we can get by *Legendre transformation*. In this case the transformation takes the following form where the total differential of the Lagrangian as a function of coordinates and velocities is:

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\, \dot{q}_i \quad ,$$

that can be written as:

$$dL = \sum_i \dot{p}_i \, dq_i + \sum_i p_i d\, \dot{q}_i \quad , \tag{1}$$

where we already know that the derivatives $\partial L / \partial \dot{q}_i$, are by definition the generalized momenta and moreover $\partial L \,/\, \partial q_i = \dot{p}_i$ by Lagrange equations. The second term in eq. (1)can be written as follows

$$\sum_i p_i d\, \dot{q}_i = d\left(\sum p_i \, \dot{q}_i\right) - \sum \dot{q}_i \, dq_i \ .$$

By taking the total diferential $d\left(\sum p_i \, \dot{q}_i\right)$ to the first term andchanging the signs we get from lagrange's equation):

$$d\left(\sum p_i \, \dot{q}_i - L\right) = -\sum \dot{p}_i \, dq_i + \sum p_i \, \dot{q}_i \quad . \tag{2}$$

The quantity under the diferential is the energy of the system as a function of the coordinates and momenta and is called *Hamiltonian function or Hamiltonian* of the system:

$$H\,(p,q,t) = \sum_i p_i\,\dot{q}_i - L\ . \tag{3}$$

Then from equ (2)

$$dH = -\sum \dot{p}_i\, dq_i + \sum p_i\, \dot{q}_i$$

where the independent variables are the coordinates and the momenta, we get the equations

$$\dot{q}_i = \frac{\partial H}{\partial p^i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}\ . \tag{4}$$

These are the equations of motion in the variables q y p and they are called Hamilton's equation.