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Subject — Physics.

Title of the topic — Motion in a central field.

Name of Program — B.Sc. (H) PHYSICS
Part-I, Paper-I (Group-B)

Two body problem Reduced to one body problem

Central force — Central force b/w two particles is one which is directed along the line joining the two particles and whose magnitude is fn. of distance r b/w them.

i.e. $\vec{F} = F(r)\,\hat{r}$.

conclusion :

1. $\vec{\tau} = 0$
2. $\vec{J} = \text{Const}$.
3. $\vec{J}$ is $\perp$ to $\vec{r}$ & $\vec{p}$ $\because$ $\vec{J} = \vec{r} \times \vec{p}$

$\Rightarrow \vec{r} \cdot \vec{J} = \vec{r} \cdot \vec{r} \times \vec{p} = 0$

$\Rightarrow$ motion of particle to a plane.

General form of Central force :
or
Inverse n^{th} Power law :

$$\vec{F} = \frac{c}{r^n}\,\hat{r} \quad \forall\ c \text{ is a const.}$$

$$\therefore -\frac{dU}{dr} = \frac{c}{r^n} \qquad U \to \text{P.E. of the particle}$$

$$\Rightarrow U = -\int \frac{c}{r^n}\,dr$$

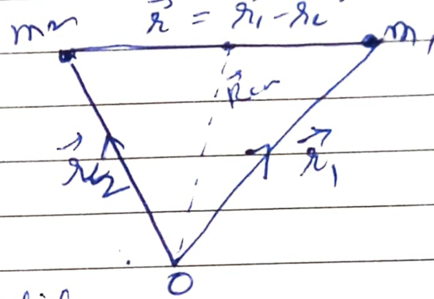
Reduction of a 2 body problem to one body problem.

9.00

Consider the motion of a system of 2 particles subjected to a central force. it is a 2 body problem. 10.00

we want to reduce it to a 1 body problem 11.00

12.00



let us consider two particles of masses m_1 & m_2 & position vectors $\vec{r}_1$ & $\vec{r}_2$ w.r.t O. 1.00

2.00

$\therefore$ central force exerted by the 2nd particle on the Ist, 3.00

$$\vec{F}_{12} = m_1\left(\frac{d^2\vec{r}_1}{dt^2}\right) \Rightarrow \frac{d^2\vec{r}_1}{dt^2} = \frac{\vec{F}_{12}}{m_1}$$ 4.00

Central force exerted by Ist particle on the 2nd,

$$\vec{F}_{21} = m_2\left(\frac{d^2\vec{r}_2}{dt^2}\right) \Rightarrow \frac{d^2\vec{r}_2}{dt} = \frac{\vec{F}_{21}}{m_2}$$ 5.00

since $\vec{F}_{12} = -\vec{F}_{21} = \vec{F} = F\hat{r}$ $\rightarrow$ $\because$ Let the central force is gravitational force. 6.00

$$\therefore \frac{d^2\vec{r}_1}{dt^2} = \frac{\vec{F}}{m_1} \quad \& \quad \frac{d^2\vec{r}_2}{dt^2} = \frac{-\vec{F}}{m_2}$$ 7.00

$$\therefore \frac{d^2\vec{r}_1}{dt^2} - \frac{d^2\vec{r}_2}{dt^2} = \frac{\vec{F}}{m_1} + \frac{\vec{F}}{m_2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}$$ 8.00

$$\frac{d^2\vec{r}_1}{dt^2} - \frac{d^2\vec{r}_2}{dt^2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{G m_1 m_2}{r^2}\hat{r}$$ 9.00

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = -\frac{1}{\mu}\frac{G m_1 m_2}{r^2}\hat{r}$$

$\left\{\because \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}\right\}$

Reduced mass of the system

$$\Rightarrow \boxed{\mu\frac{d^2\vec{r}}{dt^2} = -\frac{G m_1 m_2}{r^2}\hat{r}}$$

$\left\{\therefore \mu = \frac{m_1 m_2}{m_1 + m_2}\right.$

Since we get an Eqn. involving only one position vector $\vec{r}$ $\therefore$ It is one body problem

$\Rightarrow \frac{d^2 \vec{r}}{dt^2} = -\frac{G(m_1+m_2)}{r^2}\hat{r}$

$\Rightarrow$ It is the eqn. of motion of particle of unit mass at a vector distance $\vec{r}$ from a fixed mass (m_1+m_2)

or Compared with (m_1+m_2) ~~position of force of a direction~~

Note: $\vec{r_1}-\vec{r_2}=\vec{r}$ & $m_1\vec{r_1}+m_2\vec{r_2}=(m_1+m_2)\vec{R}_{cm}$

$\therefore \vec{r_1} = \vec{R}_{cm} + \frac{m_2\vec{r}}{m_1+m_2} = \vec{R}_{cm} + \frac{\mu\vec{r}}{m_1}$

$\vec{r_2} = \vec{R}_{cm} - \frac{m_1\vec{r}}{m_1+m_2} = \vec{R}_{cm} - \frac{m_1}{m_1+m_2}\vec{r} = \vec{R}_{cm} - \frac{\mu\vec{r}}{m_2}$

Case: 1 if $m_1 = m_2 = m$ $\therefore \mu = \frac{m}{2}$

2 if $m_1 << m_2$ $\therefore \mu = \frac{m_1 m_2}{m_2\left(\frac{m_1}{m_2}+1\right)}$

$= m_1\left(1+\frac{m_1}{m_2}\right)^{-1}$

$\simeq m_1\left(1-\frac{m_1}{m_2}\right)$

Reduced mass of H-atom $(1e^- + 1p)$ $m_p \simeq 1836\, m_e$

$\mu_H = \frac{m_p m_e}{m_p + m_e}$

$m_e << m_p$ $\therefore \mu_H \simeq m_e\left(1-\frac{m_e}{m_p}\right)$

$\simeq m_e\left(1-\frac{1}{1836}\right) \simeq m_e$.

$\therefore$ reduced mass is value of smaller mass.

Reduced mass of positronium atom (1 position + 1e) $(e^+ \,\&\, e^-)$

$\therefore \mu_p = \frac{m_e m_e}{m_e + m_e} = \frac{1}{2}m_e$

$\therefore$ Note: reduced mass of positronium atom

$= \frac{1}{2}$ of " " " Hydrogen atom

 So ~~we~~ may determine the motion of m_1 relative to m_2 by taking m_2 to lie at fixed origin of inertial frame & replace m_1 by μ. ~~we have replaced~~ exerting a force of attraction on it. ~~the eqs of~~

9.00

Angular mom. for equivalent 1 body problem

10.00

$$\vec{J} = \vec{r} \times \vec{p}$$

11.00

$$= \vec{r} \times \mu \frac{d\vec{r}}{dt} = \vec{r} \times \mu \left[\dot{r} \hat{r} + r\dot{\theta}\hat{\theta} \right]$$

12.00

$$= r\hat{r} \times \mu \left[\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \right]$$

$$= 0 + \mu r^2 \dot{\theta} \hat{z}.$$

{In cylindrical co... r, θ, z

1.00

$$\boxed{\vec{J} = \mu r^2 \omega \hat{z}}$$

2.00

3.00

In 2 body problem

$$\vec{J} = \vec{r_1} \times m_1 \frac{d\vec{r_1}}{dt} + \vec{r_2} \times m_2 \frac{d\vec{r_2}}{dt}$$

4.00

for 1B, Since $\vec{J} = \vec{r} \times \mu \frac{d\vec{r}}{dt} = (\vec{r_1} - \vec{r_2}) \times \left(\frac{m_1 m_2}{m_1 + m_2}\right)\left[\frac{d\vec{r_1}}{dt} - \frac{d\vec{r_2}}{dt}\right]$ 5.00 — (1)

w.r.t. c.m. frame of ref

$\because \vec{r} = \vec{r_1} - \vec{r_2}$

6.00

Since $\vec{R}_{cm} = \dfrac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}$

7.00

$$\Rightarrow (m_1 + m_2)\dot{\vec{R}}_{cm} = m_1\dot{\vec{r_1}} + m_2\dot{\vec{r_2}}$$

$$\Rightarrow m_1\dot{\vec{r_1}} = -m_2\dot{\vec{r_2}} \quad \left\{ \because \dot{\vec{R}}_{cm} = 0 \right.$$

8.00

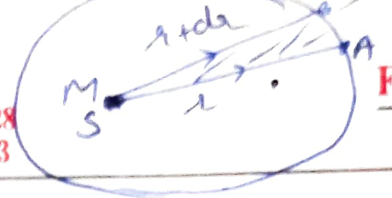
$$\Rightarrow \frac{d\vec{r_1}}{dt} = -\frac{m_2}{m_1}\frac{d\vec{r_2}}{dt} \quad \text{— (2)}$$

9.00

$$\& \quad \frac{d\vec{r_2}}{dt} = -\frac{m_1}{m_2}\frac{d\vec{r_1}}{dt} \quad \text{— (3)}$$

$\therefore$ Put (2) & (3) in (1)

$$\vec{J} = (\vec{r_1} - \vec{r_2})\left(\frac{m_1 m_2}{m_1 + m_2}\right)\left[\frac{m_1}{m_2}\frac{d\vec{r_1}}{dt} - \frac{m_2}{m_1}\frac{d\vec{r_2}}{dt}\right]$$



9.00

For the planetory motion from Δ SAB

$$\frac{\text{area Swept}}{\text{Time}} = \frac{\frac{1}{2}\vec{r}\times d\vec{r}}{dt} \quad \{\because \vec{r}\perp$$

10.00

11.00

$$\frac{d\vec{S}}{dt} = \frac{1}{2} r v$$

12.00

Since $J = mrv$ $\Rightarrow \frac{J}{m} = rv$

areal vel.

$$= \frac{1}{2}\frac{J}{m}$$

$$= \text{Constant}$$

1.00

$$= r^2\dot{\theta} \quad \{\because v = r\dot{\theta} = r\omega$$

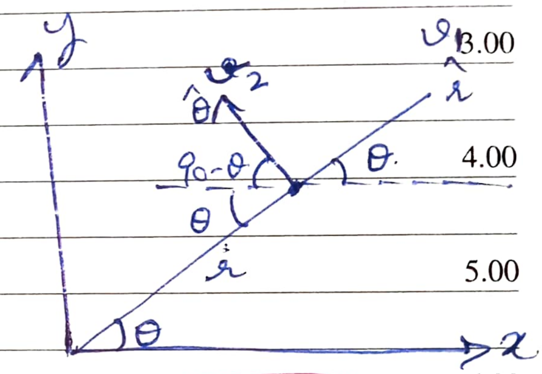
2.00

In Polar Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$\vec{V} = v_1\hat{r} + v_2\hat{\theta}$ — v_2: tangential comp., v_1: radial comp



3.00

4.00

5.00

In x dir^n.

$$v_1\cos\theta - v_2\cos(90-\theta) = \frac{dx}{dt} \quad v_x$$

6.00

$$v_1\cos\theta - v_2\sin\theta = \frac{d}{dt}(r\cos\theta) = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

7.00

$$\therefore v_1 = \dot{r} \quad \& \quad v_2 = r\dot{\theta}$$

In y dir^n

$$v_1\sin\theta - v_2\sin(90-\theta) = \frac{dy}{dt} = \frac{d}{dt}(r\sin\theta)$$

8.00

$$\Rightarrow v_1\sin\theta + v_2\cos\theta = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

9.00

$$\Rightarrow v_1 = \dot{r} \quad \& \quad v_2 = r\dot{\theta}$$

Now $\vec{a} = a_1\hat{r} + a_2\hat{\theta}$

In x dir^n

$$a_1\cos\theta - a_2\cos(90-\theta) = \frac{dv_x}{dt}$$

$$= \frac{d}{dt}\left[\dot{r}\cos\theta - r\sin\theta\,\dot{\theta}\right]$$

$$\therefore\ a_1\cos\theta - a_2\sin\theta = \ddot{r}\cos\theta - \dot{r}\sin\theta\,\dot\theta - \dot{r}\sin\theta\,\dot\theta - r\cos\theta(\dot\theta)^2 - r\sin\theta\,\ddot\theta$$

radial comp

$$\therefore\ \vec{a}_1 = \ddot{r} - r(\dot\theta)^2$$

$$\&\quad a_2 = 2\dot{r}\dot\theta + r\ddot\theta = r\ddot\theta + 2\dot{r}\dot\theta$$

Tangential comp. $= \frac{1}{r}\frac{d}{dt}(r^2\dot\theta) = 0$ $\left\{\because r^2\dot\theta = \text{const}\right.$

$$\therefore\ \vec{a} = a_1\hat{r} + a_2\hat\theta = (\ddot{r} - r\dot\theta^2)\hat{r}$$

∴ In planetary motion no tangential comp of accⁿ occurs ie total accⁿ is ~~only~~ along radial dirⁿ and no comp of force will occurs along tang. dirⁿ.

Consider the motion of particle of mass m around O m such that distance b/w them is r.

$$\therefore\ \vec{F} = -\frac{GMm}{r^2}\hat{r} = m\vec{a} = \mu\frac{d^2r}{dt^2}$$

$\because \mu = \frac{Mm}{M+m}$, $\mu \cong m$ $\because M >> m$

$$\Rightarrow -\frac{GMm}{r^2}\hat{r} = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r}$$

$$\Rightarrow \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2}$$

$$\Rightarrow r^3\frac{d^2r}{dt^2} - r^4\left(\frac{d\theta}{dt}\right)^2 = -GMr$$

{multiplying r^3 on both sides.

$$\cancel{\Rightarrow r^3\frac{d^2r}{dt^2} - r^4\left(\frac{d\theta}{dt}\right)^2}$$

$$\Rightarrow \boxed{r^3\frac{d^2r}{dt^2} - h^2 = -GMm} \quad \text{—(1)}$$

{Let $h = r^2\dot\theta = \text{const}$ & $u = \frac{1}{r}$

Now $h = r^2\dot\theta \Rightarrow \dot\theta = \frac{d\theta}{dt} = \frac{h}{r^2} = hu^2$

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{hu^2}{u^2}\frac{du}{d\theta}$$

$$\frac{dr}{dt} = -h\frac{du}{d\theta}$$

$$\therefore \frac{d^2r}{dt^2} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}$$

$$\boxed{\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2}} \quad (3)$$

$$\because h = r^2\dot{\theta} = \frac{1}{u^2}$$
$$\Rightarrow \dot{\theta} = hu^2$$

Put (3) in (1)

$$\frac{1}{u^3}\left[-h^2u^2\frac{d^2u}{d\theta^2}\right] - h^2 = -\frac{GM}{u}$$

$$\Rightarrow -\frac{h^2}{u}\frac{d^2u}{d\theta^2} - h^2 = -\frac{GM}{u}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = +\frac{GM}{h^2}$$

$$\Rightarrow \cancel{\frac{d^2}{d\theta^2}\left[u - \frac{GM}{h^2}\right] + \left[u - \frac{GM}{h^2}\right] = 0}$$

$$\Rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}} \quad (4)$$

$$P = \frac{\text{force}}{\text{mass} = m} = \frac{GM}{r^2}$$
$$\because F = -\frac{GMm}{r^2}$$

This is the ~~general~~ differential Eqn of the orbit of a particle moving under an attractive central force P per unit mass.

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{K}{h^2}} \quad (5)$$

$$\because \text{let } K = Pr^2 = \frac{P}{u^2}$$
$$K = +GM$$

General Sol^n of ~~this~~ Eqn is

$$u = \frac{K}{h^2} + A\cos(\theta - \theta_0) \quad (6)$$

A & θ_0 are constants of Eqn.

~~Now~~ let T, U & E be the kinetic, pot. & total energy per unit mass.

9.00 $\therefore T = \frac{1}{2}v^2 = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right]$

10.00 $T = \frac{1}{2}h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$

11.00 $\Rightarrow \underset{P.E.}{E - U} = \frac{1}{2}h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$

12.00

1.00 $\Rightarrow \left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-U)}{h^2}$ —— (7)

2.00 P.E per unit mass, $U = \frac{-K}{r} = -Ku$ —— (8)

3.00 Put the value of (8) & (6) in Eqn. no. (7)

4.00 $(-A\sin(\theta-\theta_0))^2 + \left(\frac{K}{h^2} + A\cos(\theta-\theta_0)\right)^2 = \frac{2}{h^2}\left[E + K\left(\frac{K}{h^2} + A\cos(\theta-\theta_0)\right)\right]$

5.00 $\Rightarrow \left[A^2\sin^2(\theta-\theta_0) + \frac{K^2}{h^4} + A^2\cos^2(\theta-\theta_0)\right] = \left[\frac{2E}{h^2} + \frac{2K^2}{h^4} + \frac{2KA\cos(\theta-\theta_0)}{h^2}\right]$

6.00 $+ 2A\frac{K}{h^2}A\cos(\theta-\theta_0)\Big]$

7.00 $\Rightarrow$ $\boxed{A^2 = \frac{K^2}{h^4} + \frac{2E}{h^2}}$ —— (9)

8.00 Putting (9) in (6) ——

9.00 $u = \frac{K}{h^2} + \sqrt{\frac{K^2}{h^4} + \frac{2E}{h^2}}\cos(\theta-\theta_0)$ let $\theta_0 = 0$

$\boxed{u = \frac{K}{h^2}\left(1 + \sqrt{1 + \frac{2Eh^2}{K^2}}\cos\theta\right)}$ —— (10)

Eqn. of Conic Section in polar Coordinate is given by

$\boxed{u = \frac{1}{l}(1 + e\cos\theta)}$ $\forall$ $l \rightarrow$ semi latus Rectum

(11)

NOTE: eccentricity = $\frac{\text{distance b/w the foci}}{\text{length of major axis}}$

i.e. half of focal chord parallel to directrix.

e = eccentricity

Comparing eqn. (10) & (11)

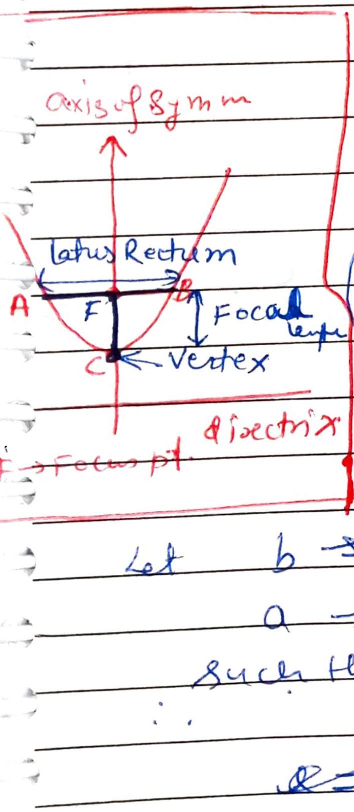
$$l = \frac{h^2}{K} \quad \& \quad e = \sqrt{1 + \frac{2Eh^2}{K^2}} \qquad (12)$$

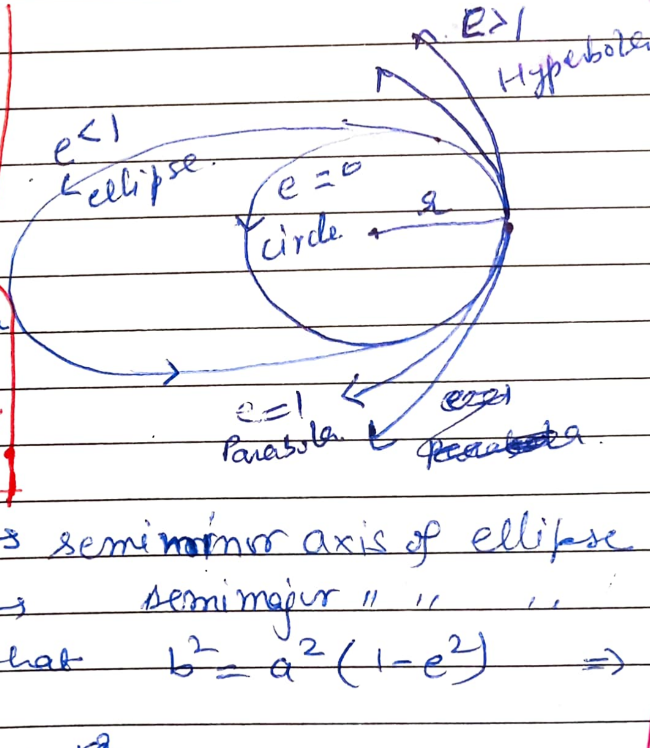
Conic section = Intersection of right circular cones by a plane.

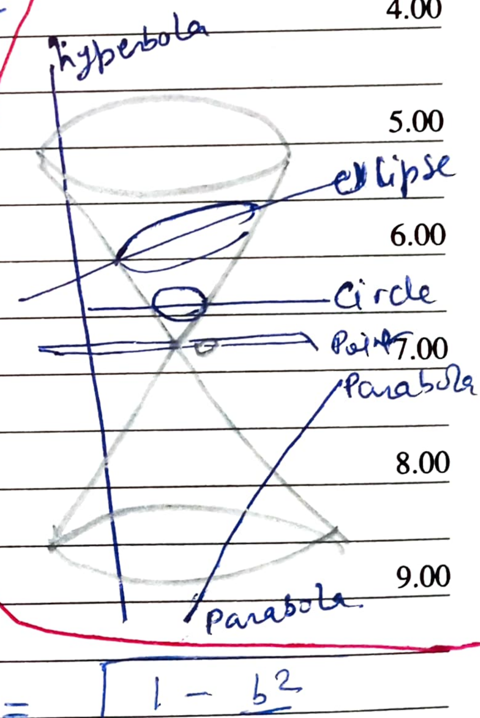
The conic may be of following type :—

(1) ellipse if $e < 1$ or $E < 0$ or −ve.
(2) parabola if $e = 1$ or $E = 0$
(3) hyperbola if $e > 1$ or $E > 0$
(4) circle if $e = 0$, or $E = \frac{-k^2}{2h^2}$

e = measure of noncircularity of an elliptical orbit.







Let $b \rightarrow$ semiminor axis of ellipse
$a \rightarrow$ semimajor " " "
such that $b^2 = a^2(1-e^2) \Rightarrow e = \sqrt{1 - \frac{b^2}{a^2}}$

$\therefore$ $l = \frac{b^2}{a}$

$$\boxed{e = \sqrt{1 - \left(\frac{l}{a}\right)}} \qquad (13) \qquad \left\{ \because l = \frac{b^2}{a} = \frac{h^2}{K} \right.$$

Comparing eqn. (12) & (13)

$l = \frac{h^2}{K}$; $a = -\frac{K}{2E}$ $\Rightarrow$ $\boxed{E = -\frac{K}{2a}}$ (14)

T.E/mass of Particle

& $h = \sqrt{Ka(1-e^2)}$

Speed of the Particle :

$\frac{\text{Energy}}{\text{mass}} = E = \underbrace{\frac{1}{2}v^2}_{KE/mass} - \underbrace{\frac{K}{r}}_{P.E/mass}$

$\Rightarrow -\frac{K}{2a} = \frac{1}{2}v^2 - \frac{K}{r}$

$\Rightarrow v^2 = K\left(\frac{2}{r} - \frac{1}{a}\right)$.

Periodic Time of the particle : $\underbrace{\frac{ds}{dt}}_{\text{areal vel.}} = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2}$

∴ Time taken by the ~~body~~ particle to ~~move~~ sweep the entire area of the ellipse.

$\tau = \frac{\text{Total } \cancel{\text{distance}} \text{ area of the ellipse}}{\text{areal vel.}}$

$\tau = \frac{\pi ab}{\frac{h}{2}} = \frac{2\pi ab}{h}$

$\Rightarrow \tau^2 = \frac{4\pi^2 a^2 b^2}{h^2}$

$= \frac{4\pi^2 a^2 a^2(1-e^2)}{Ka(1-e^2)}$

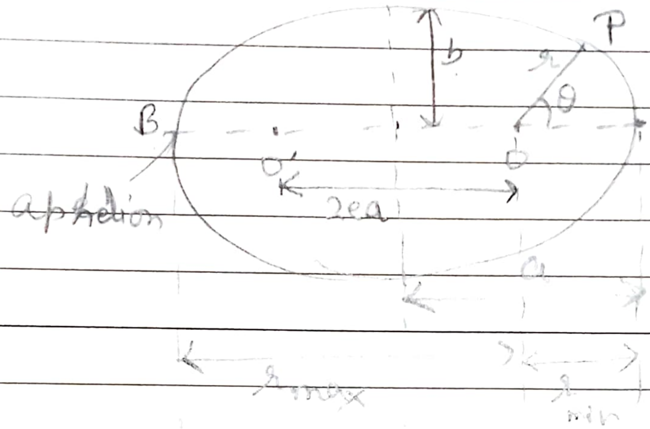
$\left\{ \begin{array}{l} b^2 = a^2(1-e^2) \\ \& \ (1-e^2) = \frac{h^2}{Ka} \end{array} \right.$

$\tau^2 = \frac{4\pi^2 a^3}{K}$ $\Rightarrow$ $\tau = 2\pi\sqrt{\frac{a^3}{K}}$

[Here $K = -GM$

$\Rightarrow \boxed{\tau^2 \propto a^3}$ (15)

$\Rightarrow$ Two planets having same semimajor axis have the same time period.



e = distance b/w the foci / length of major axis

$e = \frac{OO'}{2a}$

$\Rightarrow \boxed{OO' = 2ea}$

In general $\frac{l}{r} = 1 + e\cos\theta$. $\quad l = \frac{b^2}{a} = \frac{h^2}{K}$.

$\therefore$ at pt A $\theta = 0$ $\therefore \boxed{\frac{l}{r_{min}} = 1 + e}$ (16)

$\Rightarrow r_{min} = \frac{l}{1+e}$.

$\Rightarrow \boxed{r_{min} = \frac{b^2/a}{1+e}}$

at pt B, $\theta = 180°$ $\therefore \boxed{\frac{l}{r_{max}} = 1 - e}$ (17)

$\Rightarrow r_{max} = \frac{l}{1-e}$

$\boxed{r_{max} = \frac{b^2/a}{1-e}}$

eccentricity $e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$.

$$a = \frac{r_{max} + r_{min}}{2}$$

$\left\{ \because b^2 = a^2(1-e^2) \right.$

$l = \frac{b^2}{a}$

i.e. avg of max & min distance of the planet from centre of force i.e. the Sun.

Note: At the perihelion & Aphelion, the vel. vector is $\perp$ to the position vector $\vec{r}$.

$\therefore$ At these points $v = r\dot{\theta}$

$\left\{ \because v_1 = 0 \quad v_2 = r\dot{\theta} \right.$

$$= \frac{rh}{r^2}$$

$\left\{ \because h = r^2\dot{\theta} \right.$

$$= \frac{h}{r}$$

$h = \frac{J}{m}$

$$v = \frac{J}{mr}.$$

$\frac{ds}{dt} = \frac{1}{2} r^2 \dot{\theta}$

$= \frac{1}{2} r v$

$= \frac{1}{2} \frac{J}{m}$

$\therefore$ Total Energy of the system

$$E = \frac{1}{2} m \left(\frac{J}{mr}\right)^2 - \frac{GMm}{r}$$

$$= \frac{1}{2} \frac{J^2}{m} \left(\frac{1}{r_{min}}\right)^2 - GMm\left(\frac{1}{r_{min}}\right) \quad (18)$$

$$= \frac{1}{2} \frac{J^2}{m} \left(\frac{1}{r_{max}}\right)^2 - GMm\left(\frac{1}{r_{max}}\right) \quad (19)$$

~~Put the value of r_{min} from (16) to (18).~~

Now $l = \frac{h^2}{GM} = \frac{J^2}{GMm^2}$ $\left\{ \because h = \frac{J}{m} \right.$

$$\therefore \frac{1}{r_{min}} = \frac{(1+e)}{J^2}(GMm^2)$$

$$E = \frac{1}{2}\frac{J^2}{m}\left[\frac{(1+e)\,GMm^2}{J^2}\right]^2 - GMm\left[\frac{(1+e)\,GMm^2}{J^2}\right]$$

$$= \frac{1}{2}\frac{G^2M^2m^3}{J^2}(1+e)^2 - \frac{G^2M^2m^3}{J^2}(1+e)$$

$$= \frac{G^2M^2m^3}{J^2}(1+e)\left[\frac{1+e}{2} - 1\right]$$

$$= \frac{G^2M^2m^3}{J^2}(1+e)\left(\frac{e-1}{2}\right)$$

$$\boxed{E = -\frac{G^2M^2m^3}{2J^2}(1-e^2)}$$

for $e = 0$

$$\boxed{E = -\frac{G^2M^2m^3}{2J^2}}$$

$\Rightarrow$ if $e < 1$ The path of planet is elliptical.

NOTE: from relation derived Earlier

$$e = \sqrt{1 + \frac{2Eh^2}{k^2}}$$

$k = -GM$

$h = \frac{J}{m}$

$$\Rightarrow \frac{E}{m} = \frac{(e^2-1)\,k^2}{2h^2}$$

$$= (e^2-1)\frac{G^2M^2m^2}{2J^2}$$

$$\frac{E}{m} = -(1-e^2)\frac{G^2M^2m^2}{2J^2}$$

$$\Rightarrow E = -(1-e^2)\frac{G^2M^2m^3}{2J^2}$$

In General $E = -\frac{(1-e^2)\,G^2M^2m^2\mu}{2J^2}$

$\forall\ \mu$ = reduced mass of the ~~gas~~ system