Topic 1: Least action principle and Lagranges Equation, B.Sc. (Hons.) Part -1, Paper 1

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The least action principle and Lagranges Equation

The general formulation of the law of motion of mechanical systems is the *action or Hamilton principle*. According to this principle every mechanical system is characterized by a function defined as:

$$L\left(q_1, q_2, ..., q_s, \dot{q}_1, \dot{q}_2, \dot{q}_s, t\right),$$

or $L \left(q, \dot{q}, t\right)$, and the motion of the system satisfies the following condition: let at the moments t_1 and t_2 the system is in the positions given by the set of coordinates $q^{(1)}$ y $q^{(2)}$; the system moves between these positions in such a way that the integral

$$S = \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt \tag{1}$$

takes the minimum possible value. The function L is called the *Lagrangian* of the system, and the integral (1) is known as least *action* of system. The Lagrange function contains only q and $\dot{q}$, and no other higher-order derivatives, as the mechanical state is completely defined by its coordinates and velocities.

Let $q = q(t)$ be the function for which S is a minimum. This means that S grows when one $q(t)$ is replaced by an arbitrary function

$$q(t) + \delta q(t), \tag{2}$$

where $\delta q(t)$ is a small function through the interval from t_1 to t_2 [it is called the variation of the function $q(t)$]. Since at t_1 and t_2 all the functions (2) should take the same values $q^{(1)}$ and $q^{(2)}$, we get:

$$\delta q\left(t_1\right)=\delta q\left(t_2\right)=0. \tag{3}$$

we may change S when q is replaced by $q+\delta q$ is given by:

$$\int_{t_1}^{t_2} L\left(q+\delta q, \dot{q}+\delta \dot{q}, t\right) dt-\int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt.$$

The necessary condition of minimum (or, in general, extremum) for S is that the sum of all terms turns to zero; Thus, the action principle can be written as :

$$\delta S=\delta \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt=0, \tag{4}$$

or we may write : $\int_{t_2}^{t_1}\left(\frac{\partial L}{\partial q} \delta q+\frac{\partial L}{\partial \dot{q}} \delta \dot{q}\right) dt=0$.

Considering $\delta \dot{q}= d/dt\,(\delta q)$, we get

$$\delta S=\left[\frac{\partial L}{\partial \dot{q}} \delta q\right]_{t_1}^{t_2}+\int_{t_2}^{t_1}\left(\frac{\partial L}{\partial q}-\frac{d}{dt} \frac{\partial L}{\partial \dot{q}}\right) \delta q dt=0 \ . \tag{5}$$

Considering the conditions (3), the first term of this expresion vanishes. Only the integral remains that should be zero for all values of δq. This is possible only if the integrand is zero, which gives:

$$\frac{\partial L}{\partial q}-\frac{d}{dt} \frac{\partial L}{\partial \dot{q}}=0 \ .$$

For more degrees of freedom, the s different functions $q_i(t)$ should vary independently. Thus, it is obvious that we get s equations of the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0 \qquad (i=1,2,...,s) \tag{6}$$

These are the equations are called *Lagrange equations*. If the Lagrangian for given mechanical system is known, then the equations (6) form the relationship between the accelerations, the velocities and the coordinates; or, they are the equations of motion of the system.

By knowing them completely we define the movement of the mechanical system. To get this, it is necessary to know the initial conditions that characterize the state of the system at a given moment , example, the initial values of the coordinates and velocities.