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Exact differential equation
(Integrating factor)

Integrating factor:→ If the equation $Mdx + Ndy = 0$ is not exact, it can always be made exact by multiplying some function of x and y. such a multiplier is called integrating factor.

Note:- Often an integrating factor (I.F) can be found out by inspection.

Some exact differential are listed below.

(i) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$ (ii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iii) $d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$ (iv) $d\left(\frac{x^2}{y}\right) = \frac{2xy\,dx - x^2dy}{y^2}$

(v) $d(\log xy) = \frac{xdy + ydx}{xy}$ (vi) $d(xy) = xdy + ydx$

(vii) $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{xdy - ydx}{x^2+y^2}$ (viii) $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{ydx - xdy}{x^2+y^2}$

(ix) $d\left(\log \frac{y}{x}\right) = \frac{xdy - ydx}{xy}$ (x) $d\left(\log \frac{x}{y}\right) = \frac{ydx - xdy}{xy}$

Rule I: If the equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$, then $1/(Mx+Ny)$ is an integrating factor.

Ex! Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Sol: Clearly given equation is homogeneous, having

$M = x^2y - 2xy^2$ and $N = -(x^3 - 3x^2y)$

Now $Mx + Ny = x(x^2y - 2xy^2) - y(x^3 - 3x^2y)$
$= x^2y^2 \neq 0$

Thus $\frac{1}{x^2y^2}$ is an integrating factor.

Multipling given equation by $\frac{1}{x^2y^2}$, we have

$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$, which is an exact.

∴ solution is

$$\int_{y=c}\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy = 0$$

$\Rightarrow \frac{x}{y} - \log x^2 + \log y^3 = \log c$

$\log \frac{y^3}{cx^2} = -x/y$ ∴ $y^3 = cx^2e^{-x/y}$

Rule II: If the equation $Mdx + Ndy = 0$ is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$, then $1/(Mx - Ny)$ is an integrating factor, provided $Mx - Ny \neq 0$

EX: $y(1+xy)dx + x(1-xy)dy = 0$

Sol: Here given equation is

$y(1+xy)dx + x(1-xy)dy = 0$ is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$

Now $M = y(1+xy)$ and $N = x(1-xy)$

$$Mx - Ny = xy(1+xy) - xy(1-xy)$$
$$= 2x^2y^2 \neq 0$$

Thus I.F is $1/2x^2y^2$

Multiplying $1/2x^2y^2$ in given equation, we have

$\frac{1}{2}\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \frac{1}{2}\left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$, which is exact and therefore its solution is

$$\int_{y=c}\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int\left(-\frac{1}{2y}\right)dy = c$$

$$\Rightarrow -\frac{1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\log\left(\frac{x}{y}\right) = c + \frac{1}{2xy}$$

$$\frac{x}{y} = e^{c} \cdot e^{1/2xy}$$

$$\frac{x}{y} = c' e^{1/2xy} \Rightarrow x = yc'e^{1/2xy}.$$

Rule III :→ If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x along say $f(x)$, then $e^{\int f(x)dx}$ is an integrating factor of $Mdx + Ndy = 0$

Rule IV :→ If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is function of y alone say $f(y)$, then $e^{\int f(y)dy}$ is an integrating factor.

EX: Solve $(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3 + y^2)dy = 0$

Sol:→ Here given equation is

$(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3 + y^2)dy = 0$

$M = xy^2 - x^2$ and $N = 3x^2y^2 + x^2y - 2x^3 + y^2$

$\frac{\partial M}{\partial y} = 2xy$ $\qquad \frac{\partial N}{\partial x} = 6xy^2 + 2xy - 6x^2$

Now $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{xy^2-x^2}\left\{6xy^2+2xy-6x^2-2xy\right\}=\frac{6x(y^2-x)}{x(y^2-x)}=6$
$=f(y)$

$\therefore$ I.F $= e^{\int 6\,dy} = e^{6y}$

Multiply, given differential equation by e^{6y}, we have.

$e^{6y}(xy^2-x^2)dx + e^{6y}(3x^2y^2+x^2y-2x^3+y^2)dy=0$, which is exact

$\therefore$ Solution is

$$\int_{y=c} e^{6y}(xy^2-x^2)dx + \int e^{6y}y^2dy = c$$

On simplification we have

$$e^{6y}\left(\frac{1}{2}x^2y^2-\frac{1}{3}x^3+\frac{1}{6}y^2-\frac{1}{18}y+\frac{1}{108}\right)=c$$

Rule V :→ If the equation $Mdx+Ndy=0$ can be put in the form $x^\alpha y^\beta(my\,dx+nx\,dy)+x^{\alpha'}y^{\beta'}(m'y\,dx+n'x\,dy)=0$, where $\alpha,\beta,m,n,\alpha',\beta',m',n'$ are constants, then the given equation has an I.F. x^hy^k where h and k are obtained by applying condition exact after multiplication with x^hy^k.

Ex. Solve $(y^2+2x^2y)dx+(2x^3-xy)dy=0$

Sol: Given equation is $(y^2+2x^2y)dx+(2x^3-xy)dy=0$ — (1)

The equation (1) can be written as

$$y(y\,dx-x\,dy)+x^2(2y\,dx+2x\,dy)=0 \quad -(2)$$

Which is of the form $x^\alpha y^\beta(my\,dx+nx\,dy)+x^{\alpha'}y^{\beta'}(m'y\,dx+n'x\,dy)=0$

Multiply (2) by x^hy^k, we have

$$(x^hy^{k+2}+2x^{h+2}y^{k+1})dx+(2x^{h+3}y^k-x^{h+1}y^{k+1})dy=0 \quad -(3)$$

Compare with $Mdx+Ndy=0$, we have

$M=x^hy^{k+2}+2x^{h+2}y^{k+1}$, $N=2x^{h+3}y^k-x^{h+1}y^{k+1}$

Since (3) exact, we must have $\partial M/\partial y=\partial N/\partial x$

i.e. $(k+2)x^hy^{k+1}+2(k+1)x^{h+2}y^k=2(h+3)x^{h+2}y^k-(h+1)x^hy^{k+1}$

Equating the coeffs of x^hy^{k+1} and $x^{h+2}y^k$

$(k+2)=-(h+1)$ and $2(k+1)=2(h+3)$

i.e. $h+k=-3$ and $h-k=-2$

On solving we get $h=-5/2$, $k=-1/2$

Thus I.F = $x^{-5/2}y^{-1/2}$, Multiplying (1) with $x^{-5/2}y^{-1/2}$ we get

$$(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2})dx + (2x^{1/2}y^{-1/2}+x^{-3/2}y^{1/2})dy=0$$

which exact. Thus its solution is

$$\int_{y=c}(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2})dx = c$$

$$\Rightarrow \quad -\tfrac{2}{3}x^{-3/2}y^{3/2}+4x^{1/2}y^{1/2} = c$$

Solve the following differential equation

(i) $e^y dx + (xe^y+2y)dy=0$ Ans: $xe^y+y^2=c$

(ii) $(x^2y^2+xy+1)y\,dx + (x^2y^2-xy+1)x\,dy=0$ Ans: $xy-(1/xy)+\log(x/y)=c$

(iii) $(xy^2+2x^2y^3)dx + (x^2y-x^3y^2)dy=0$ Ans: $\log(x^2/y)-(1/xy)=c$

(iv) $(x^2+y^2+2x)dx + 2y\,dy=0$ Ans $e^x(x^2+y^2)=c$

(v) $(x^3-2y^2)dx + 2xy\,dy=0$ Ans: $x+(y^2/x^2)=c$

(vi) $(x^2+y^2+1)dx + x(x-2y)dy=0$ Ans: $x+y-(y^2+1)/x=c$

(vii) $(xy^3+y)dx + 2(x^2y^2+x+y^4)=0$ Ans: $3x^2y^4+6xy^2+2y=c$

(viii) $(2xy^2-2y)dx + (3x^2y-4x)dy=0$ Ans: $x^2y^3+2xy^2=c$

(ix) $(y^4+2y)dx + (xy^3+2y^4-4x)dy=0$ Ans: $x(y+2/y^2)+y^2=c$

(x) $(4xy^2+6y)dx + (5x^2y+8x)dy=0$ Ans: $x^4y^5+2x^3y^4=c$