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Differential equations and their classification

Differential equation:→ An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variable is called a differential equation.

Examples:

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0 \qquad (1)$$

$$dy = (x+\sin x)\,dx \qquad (2)$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \qquad (3)$$

Ordinary differential equation :→ A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.
Equation (1) and (2) are ordinary differential equations.

Partial differential equation :→ A differential equation involving partial derivatives of one or more independent variables is called a partial differential equation.
Equation (3) is example of partial differential equation.

Order of a differential equation :→ The order of the highest order derivatives involved in a differential equation is called the order of the differential equation.
Equation (2) and (3) are of order first and equation

Linear differential equation :→ A differential equation is called linear if

(i) every dependent variable and its various derivatives occurs to the first degree.

(ii) No products of dependent variables and (or) derivatives occur.

Equation (2-3) are example of linear differential, while (1) is non linear differential equation.

Solution of a differential equation :→ Any relation between the dependent and independent variables, when substituted in the differential equation, reduces to an identity is called a solution of the differential equation.

Consider the nth order ordinary differential equation

$$F\left[x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right] = 0 \qquad -(4)$$

Where F is a real function of its (n+1) arguments, x, y, $\frac{dy}{dx}, \ldots \frac{d^n y}{dx^n}$.

(a) A solution of (4) containing n independent arbitrary constants is called a general solution.

(b) A solution of (4) obtained from general solution of (4) by giving particular values to one or more of the n independent arbitrary constants is called a particular solution of (4).

(c) A solution of (4) which cannot be obtained from any general solution of (4) by any choice of the n independent arbitrary constant is called a singular solution of (4)

For example, consider differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \qquad -(5)$$

The general solution of (5) is

$$y = c_1 e^{x} + c_2 e^{2x} \qquad -(6)$$

The particular solutions can be found as by assigning particular values.

$$y = 3e^{x} + 4e^{2x}, \quad y = 4e^{x} + 6e^{2x}.$$

Further it can by verified as $y = (x+c)^2$ is ~~also~~ general solution of differential equation ~~(5), known as singular solution~~

The equation $y = (x+c)^2$ is general solution of the equation $\left(\frac{dy}{dx}\right)^2 = 4y$. ——— (7)

Also $y=0$ is also solution of eq (7), which cannot be obtained by any choice of c. Hence $y=0$ is a singular solution.

Exact differential equation :→

The expression $M(x,y)\,dx + N(x,y)\,dy = 0$ ——— (8)
is called exact differential equation, if there exists a function $F(x,y)$, such that F has its first partial derivation as

$$d[F(x,y)] = M\,dx + N\,dy$$

i.e. $\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = M\,dx + N\,dy$.

Ex: The differential equation $y^2 dx + 2xy\,dy = 0$ is an exact differential equation, as there exist function xy^2 such that

$$d(xy^2) = \frac{\partial}{\partial x}(xy^2)dx + \frac{\partial}{\partial y}(xy^2)dy$$
$$= y^2 dx + 2xy\,dy.$$

Statement: The necessary and sufficient condition for differential equation $M\,dx + N\,dy = 0$, to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof:→ Let the differential equation is exact.
Hence by defination, there exist function $F(x,y)$ such that

$$d[F(x,y)] = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = M\,dx + N\,dy$$

$\Rightarrow$ $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$

Now $\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}$

since $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$ [∵ since F has its partial derivatives]

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Conversely, suppose condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exist for a differential equation $Mdx + Ndy = 0$.

Now, we have to find a function $F(x,y)$, such that

$$d(F(x,y)) = Mdx + Ndy$$

i.e. $\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = Mdx + Ndy$

i.e. $\frac{\partial F}{\partial x} = M(x,y)$ — (9) and $\frac{\partial F}{\partial y} = N(x,y)$ — (10)

Now $\frac{\partial F(x,y)}{\partial x} = M(x,y)$

$$\Rightarrow \quad F(x,y) = \int M(x,y)\partial x + \phi(y) \qquad — (11)$$

where $\int M(x,y)\partial x$ indicates partial integration w.r.t x keeping y constant. Differentiating (11) partially w.r.t y

$$\frac{\partial F(x,y)}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,\partial x + \frac{d\phi(y)}{dy} \qquad — (12)$$

Also from (10) and (12), we have

$$N(x,y) = \frac{\partial}{\partial y}\int M(x,y)\,\partial x + \frac{d\phi(y)}{dy}$$

$$\Rightarrow \quad \frac{d\phi(y)}{dy} = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,\partial x \qquad — (13)$$

Since ϕ is function of y only, $\frac{d\phi}{dy}$ must also be independent of x

Now we show that $\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,\partial x\right] = 0$

$$\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,\partial x\right] = \left[\frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x \partial y}\int M(x,y)\,\partial x\right]$$

$$= \left[\frac{\partial N}{\partial x} - \frac{\partial^2}{\partial y \partial x}\int M(x,y)\,\partial x\right]$$

$$= \left[\frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\int M(x,y)\,\partial x\right]\right]$$

$$= \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

$$= 0$$

Thus from (13)

$$\phi(y) = \int\left[N(x,y) - \int\frac{\partial M}{\partial y}\partial x\right]dy$$

∴ By (11), we have

$$F(x,y) = \int M(x,y)\,\partial x + \int\left[N(x,y) - \int\frac{\partial M}{\partial y}\partial x\right]dy$$

Since $F(x,y)$ satisfy the conditions ∴ $Mdx + Ndy = 0$ is exact.

Ex. Solve the equation $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$

Sol: comparing the equation with $Mdx+Ndy=0$, we have

$M = x^2-4xy-2y^2$, $N = y^2-4xy-2x^2$

$\therefore \dfrac{\partial M}{\partial y} = -4x-4y$ $\qquad \dfrac{\partial N}{\partial x} = -4y-4x$

Clearly $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Hence, the given equation is exact and hence its solution is

$\int M\,dx$ [Treating y as constant] $+ \int (\text{terms of } N \text{ not containing } x)\,dy = c'$

$\therefore \int (x^2-4xy-2y^2)\,dx + \int y^2\,dy = c'$

$\Rightarrow \dfrac{x^3}{3} - 2x^2y - 2xy^2 + \dfrac{y^3}{3} = c'$

$\Rightarrow x^3+y^3-6xy(x+y) = c$ where $c = 3c'$.

Ex: Solve $(r+\sin\theta-\cos\theta)dr + r(\sin\theta+\cos\theta)d\theta = 0$

Sol:→ comparing the given equation with $Mdr+Nd\theta=0$

$M = r+\sin\theta-\cos\theta$ and $N = r(\sin\theta+\cos\theta)$

$\dfrac{\partial M}{\partial\theta} = \cos\theta+\sin\theta$ and $\dfrac{\partial N}{\partial r} = \sin\theta+\cos\theta$

Thus $\dfrac{\partial M}{\partial\theta} = \dfrac{\partial N}{\partial r}$. so equation is exact with solution

$\int\limits_{\theta=c} M\,dr + \int (\text{terms of } N \text{ not containing } r)\,d\theta = c$

$\Rightarrow \int (r+\sin\theta-\cos\theta)\,dr = c$

$\Rightarrow \dfrac{r^2}{2} + r(\sin\theta-\cos\theta) = c$

Solve the following Equation

(i) $(x+2y-2)dx+(2x-y+3)dy=0$ — Ans: $x^2+4xy-4x-y^2+6y=c$

(ii) $(2ax+by)y\,dx+(ax+2by)x\,dy=0$ — Ans: $ayx^2+by^2x=c$

(iii) $(3x^2+4xy)dx+(2x^2+2y)dy=0$ — Ans: $x^3+2x^2y+y^2=c$

(iv) $(e^y+1)\cos x\,dx+e^y\sin x\,dy=0$ — Ans: $(e^y+1)\sin x=c$