Topic 2: D'Alembert principle, Virtual work B.Sc. (Hons.) Part -1, Paper 1

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D'Alembert principle

The virtual displacement of a system is the change in its configurational space under an arbitrary infinitesimal variation of the coordinates $\delta\mathbf{r}_i$, *which is compatible with the forces and constraints imposed on the system at the given instant t.* It is called virtual in order to distinguish it from the real one, which takes place in a time interval dt, during which the forces and the constraints can vary.

The constraints introduce two types of conditions:

(1) Not all the coordinates are independent.

(2) In general, the constraint forces are not known, they are some unknowns of the problem and they should be obtained from the solution .

In the case of holonomic constraints we introduc a set of independent coordinates ($q_{1,}q_{2,...,}q_m$, where m is the number of degrees of freedom involved). This means that if there are m constraint equations and $3N$ coordenates $(x_1, ..., x_{3N})$, we can eliminate these n equations by introducing the independent variables $(q_1, q_2, .., , q_n)$. A transformation of the following form is used

$$x_1 = f_1(q_1, ..., q_m, t)$$

$$\vdots$$

$$x_{3N} = f_{3N}(q_1, ..., q_n, t) \ ,$$

where $n = 3N - m$.

For simplicity, in (2) Mechanical system needs to be formulated in such a way that the forces of constraint *do not occur* in the solution of the problem.

Virtual work: We assume that a system of N particles is described by $3N$ coordenates $(x_1, x_2, ..., x_{3N})$ and let $F_1, F_2, ..., F_{3N}$ be the components of the forces acting on each particle. If the particles of the system is for infinitesimal and instantaneous displacements $\delta x_1, \delta x_2, ..., \delta x_{3N}$ under the action of the $3N$ forces, then the performed work is:

$$\delta W = \sum_{j=1}^{3N} F_j \delta x_j \ . \tag{1}$$

Such displacements are known as *virtual displacements* and δW is called *virtual work*; (1) can be also written as:

$$\delta W = \sum_{\alpha=1}^{N} \mathbf{F}_\alpha \cdot \delta \mathbf{r} \ . \tag{2}$$

Forces of constraint: besides the applied forces $\mathbf{F}_\alpha^{(e)}$, the particles can be acted on by forces of constraint $\mathbf{F}_\alpha$.

The principle of virtual work: Let $\mathbf{F}_\alpha$ be the force acting on the particle α of the system. If we separate $\mathbf{F}_\alpha$ in a contribution from the outside $\mathbf{F}_\alpha^{(e)}$ and the constraint $\mathbf{R}_\alpha$

$$\mathbf{F}_\alpha = \mathbf{F}_\alpha^{(e)} + \mathbf{R}_\alpha \ . \tag{3}$$

and if the system is in equilibrium, then

$$\mathbf{F}_\alpha = \mathbf{F}_\alpha^{(e)} + \mathbf{R}_\alpha = 0 \ . \tag{4}$$

Thus, the virtual work due to all possible forces $\mathbf{F}_\alpha$ is:

$$W = \sum_{\alpha=1}^{N} \mathbf{F}_\alpha \cdot \delta \mathbf{r}_\alpha = \sum_{\alpha=1}^{N} \left(\mathbf{F}_\alpha^{(e)} + \mathbf{R}_\alpha \right) \cdot \delta \mathbf{r}_\alpha = 0 \ . \tag{5}$$

If the system is such that the constraint forces do not make virtual work, then from (5) we obtain:

$$\sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{(e)} \cdot \delta \mathbf{r}_\alpha = 0 \ . \tag{6}$$

According to Newton, the equation of motion is:

$$\mathbf{F}_\alpha = \dot{\mathbf{p}}_\alpha$$

and can be written in the form

$$\mathbf{F}_\alpha - \dot{\mathbf{p}}_\alpha = 0 \ ,$$

which means that the particles of the system would be in equilibrium under the action of a force equal to the real one plus an inverted force $-\dot{\mathbf{p}}_i$. (6) we can write

$$\sum_{\alpha=1}^{N} \left(\mathbf{F}_\alpha - \dot{\mathbf{p}}_\alpha \right) \cdot \delta \mathbf{r}_\alpha = 0 \tag{7}$$

and by solving, we obtain:

$$\sum_{\alpha=1}^{N} \left(\mathbf{F}_\alpha^{(e)} - \dot{\mathbf{p}}_\alpha \right) \cdot \delta \mathbf{r}_\alpha + \sum_{\alpha=1}^{N} \mathbf{f}_\alpha \cdot \delta \mathbf{r}_\alpha = 0 \ .$$

Again, let us limit ourselves to systems for which the virtual work due to the forces of constraint is zero leading to

$$\sum_{\alpha=1}^{N} \left(\mathbf{F}_\alpha^{(e)} - \dot{\mathbf{p}}_\alpha \right) \cdot \delta \mathbf{r}_\alpha = 0 \ , \tag{8}$$

which is the *D'Alembert's principle*. This equation does not have a useful form . Therefore, we should change the principle to, virtual displacements of the generalized coordinates, which being independent from each other, imply zero coefficients for $\delta \mathbf{q}_\alpha$. Thus, the velocity in terms of the generalized coordinates :

$$\mathbf{v}_\alpha = \frac{d\mathbf{r}_\alpha}{dt} = \sum_k \frac{\partial \mathbf{r}_\alpha}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_\alpha}{\partial t} \qquad \text{where} \qquad \mathbf{r}_\alpha = \mathbf{r}_\alpha \left(q_1, q_2, ..., q_n, t \right) \ .$$

In same way, the relation between arbitrary virtual displacement $\delta \mathbf{r}_\alpha$ to the virtual displacements $\delta \mathbf{q}_j$ through

$$\delta \mathbf{r}_\alpha = \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \delta q_j \ .$$

Then, the virtual work $\mathbf{F}_\alpha$ expressed in terms of the generalized coordinates will be:

$$\sum_{\alpha=1}^{N} \mathbf{F}_\alpha \cdot \delta \mathbf{r}_\alpha = \sum_{j,\alpha} \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j \ , \tag{9}$$

where the Q_j are the so-called components of the generalized force, defined in the form

$$Q_j = \sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \ .$$

Now from eq. (8) :

$$\sum_\alpha \dot{\mathbf{p}} \cdot \delta \mathbf{r}_\alpha = \sum_\alpha m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \delta \mathbf{r}_\alpha \tag{10}$$

and by substituting in (10) the above result 10 can be written:

$$\sum_\alpha \left\{ \frac{d}{dt} \left(m_\alpha \mathbf{v}_\alpha \cdot \frac{\partial \mathbf{v}_\alpha}{\partial \dot{q}_j} \right) - m_\alpha \mathbf{v}_\alpha \cdot \frac{\partial \mathbf{v}_\alpha}{\partial q_j} \right\} = \sum_j \left[\left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} - Q_j \right] \delta q_j = 0 \ . \tag{11}$$

The variables q_j can be an arbitrary system of coordinates describing the motion of the system and if the constraints are holonomic, it is possible to find systems of independent coordinates q_j

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q_\alpha} = Q_j \ . \tag{12}$$

There are m equations. The equations (12) called the La-grange equations.

$$\mathbf{F}_\alpha = -\nabla_i V.$$

Then Q_j can be written as:

$$Q_j = -\frac{\partial V}{\partial q_j} \ .$$

The equations (12) can also be written in the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_j}}\right) - \frac{\partial (T-V)}{\partial q_j} = 0 \tag{13}$$

and defining the *Lagrangian* L in the form $L = T - V$ we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_j}}\right) - \frac{\partial L}{\partial q_j} = 0 \ . \tag{14}$$

These are the **Lagrange equations of motion**.