Topic 4: Conservation laws for Energy Momentum and Angular Momentum, B.Sc. (Hons.) , Part -1, Paper 1

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Conservaton laws for Energy, Momentum and Angular Momentum 1 Energy

Let us consider the conservation theorem resulting from the *homogeneity of time*. Because of this homogeneity, the Lagrangian of a closed system does not depend explicitly on time. Then, the total time diferential of the Lagrangian (not depending explicitly on time) can be written:

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$$

and according to the Lagrange equations we can rewrite the previous equation as follows:

$$\frac{dL}{dt} = \sum_i \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_i \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i = \sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) \ ,$$

or

$$\sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0 \ .$$

We can get

$$E \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \qquad (1)$$

remains constant during the movement of the closed system, that is it is an integral of motion. This constant quantity is called the *energy* E of the system.

2 Momentum

The *homogeneity of space* implies another conservation theorem. Because of this homogeneity, the mechanical properties of a closed system do not vary under a parallel displacement of the system as a whole through space. We consider an infinitesimal displacement ϵ (i.e., the position vectors $\mathbf{r}_\alpha$ to $\mathbf{r}_a + \epsilon$) and for the condition for which the Lagrangian does not change. The variation of the function L resulting from the infinitesimal change of the coordinates (maintaining constant the velocities of the particles) is given by:

$$\delta L = \sum_a \frac{\partial L}{\partial \mathbf{r}_a} \cdot \delta \mathbf{r}_a = \epsilon \cdot \sum_a \frac{\partial L}{\partial \mathbf{r}_a} ,$$

extending the sum over all the particles of the system. Since ϵ is arbitrary, the condition $\delta L = 0$ is equivalent to

$$\sum_a \frac{\partial L}{\partial \mathbf{r}_a} = 0 \tag{2}$$

and from the equations of Lagrange

$$\sum_a \frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{v}_a} \right) = \frac{d}{dt} \sum_a \frac{\partial L}{\partial \mathbf{v}_a} = 0 .$$

Thus, we get the conclusion that for a closed mechanical system the *momentum*

$$\mathbf{P} \equiv \sum_a \frac{\partial L}{\partial \mathbf{v}_a}$$

remains constant during the motion.

3 Angular momentum

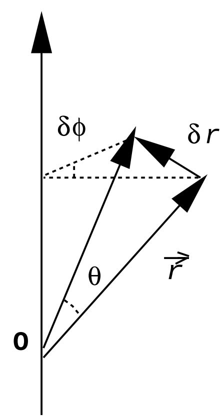
Let us study now the conservation theorem from *the isotropy of space*. For this we consider an infinitesimal rotation of the system and look for the condition under which the Lagrangian does not change.

We shall take an infinitesimal rotation vector $\delta\phi$ a vector of modulus equal to the angle of rotation $\delta\phi$ and whoose direction coincide with that of the rotation axis. Let us first consider the increment of the position vector of a particle in the system, by taking the origin of coordinates on the axis of rotation. The lineal displacement of the position vector as a function of angle is

$$|\delta \mathbf{r}| = r \sin\theta \delta\phi \ ,$$

(as in figure). The direction of the vector $\delta\mathbf{r}$ is perpendicular to the plane defined by $\mathbf{r}$ and $\delta\phi$, and therefore,

$$\delta\mathbf{r} = \delta\phi \times \mathbf{r} \ . \qquad (3)$$



The rotation of the system changes not only the directions of the position vectors but also the velocities of the particles that are modified by the same rule for all the vectors. The velocity increment with respect to a fixed frame system will be:

$$\delta\mathbf{v} = \delta\phi \times \mathbf{v} \ .$$

Let us apply these expressions to the condition that the Lagrangian does not vary under rotation:

$$\delta L = \sum_a \left(\frac{\partial L}{\partial \mathbf{r}_a} \cdot \delta\mathbf{r}_a + \frac{\partial L}{\partial \mathbf{v}_a} \cdot \delta\mathbf{v}_a \right) = 0$$

and let us substitute the definitions of the derivatives $\partial L/\partial\mathbf{v}_a$ por $\mathbf{p}_a$ and $\partial L/\partial\mathbf{r}_a$ from the Lagrange equations by $\dot{\mathbf{p}}_a$; we get

$$\sum_a \left(\dot{\mathbf{p}}_a \cdot \delta\phi \times \mathbf{r}_a + \mathbf{p}_a \cdot \delta\phi \times \mathbf{v}_a \right) = 0 \ ,$$

or by circular permutation of the factors and getting $\delta\phi$ out of the sum:

$$\delta\phi \sum_a \left(\mathbf{r}_a \times \dot{\mathbf{p}}_a + \mathbf{v}_a \times \mathbf{p}_a \right) = \delta\phi \cdot \frac{d}{dt} \sum_a \mathbf{r}_a \times \mathbf{p}_a = 0 \ ,$$

because $\delta\phi$ is arbitrary, one gets

$$\frac{d}{dt} \sum_a \mathbf{r}_a \times \mathbf{p}_a = 0$$

Thus, we conclude that during the motion of a closed system the called the *angular (or kinetic) momentum* is conserved.

$$M \equiv \sum_a \mathbf{r}_a \times \mathbf{p}_a \ .$$