TOPIC 5: Application of the least Action Principle B.Sc. (Hons.), Part -1, Paper 1

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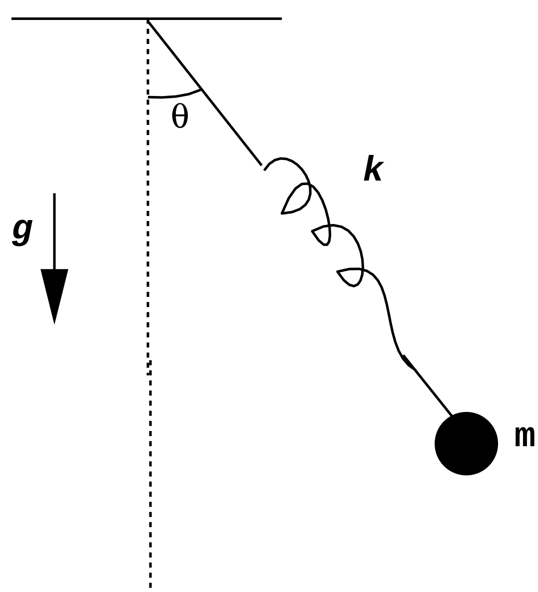
Applications of the least action principle

Equations of motion

Find the eqs of motion for a pendular mass sustained by a resort, by directly applying Hamilton's principle.



For the pendulum the Lagrangian function is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\theta}^2) + mgr\cos\theta - \frac{1}{2}k(r-r_o)^2 \ ,$$

therefore

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \left[m \left(\dot{r}\, \delta\, \dot{r} + r\, \dot{\theta}^2 + r^2\, \dot{\theta}\, \delta\, \dot{\theta} \right) + mg\delta r \cos\theta - mgr\delta\theta \sin\theta - k(r-r_o)\delta r \right] dt$$

$$m\, \dot{r}\, \delta\, \dot{r}\ dt = m\, \dot{r}\ d(\delta r) = d\left(m\, \dot{r}\ \delta r\right) - m\delta r\, \ddot{r}\ dt \ .$$

In the same way

$$mr^2\theta^2\delta\, \dot{\theta}\ dt = d\left(mr^2\, \dot{\theta}\, \delta\, \dot{\theta}\right) - \delta\theta \frac{d\left(mr^2\, \dot{\theta}\right)}{dt} dt$$

$$= d\left(mr^2\, \dot{\theta}\, \delta\, \dot{\theta}\right) - \delta\theta \left(mr^2\, \ddot{\theta} + 2mr\, \dot{r}\dot{\theta}\right) dt \ .$$

Therefore, the previous integral can be written

$$\int_{t_1}^{t_2} \left[\left\{ m\, \ddot{r} - mr\, \dot{\theta}^2 - mg\cos\theta + k\left(r - r_o\right) \right\} + \left\{ mr^2\, \ddot{\theta} + 2mr\, \dot{r}\dot{\theta} + mgr\sin\theta \right\} \delta\theta \right] dt$$

$$- \int_{t_1}^{t_2} \left[d\left(m\, \dot{r}\ \delta r\right) + d\left(mr^2\theta^2\, \dot{\theta}\ \delta\theta\right) \right] = 0 \ .$$

Assuming that both δr and $\delta\theta$ are equal zero at t_1 and t_2, the second integral is vanishes. Since δr and $\delta\theta$ are completely independent of each other, the first integral can be zero only if

$$m\, \ddot{r} - mr\, \dot{\theta}^2 - mg\cos\theta + k(r - r_o) = 0$$

and

$$mr^2\, \ddot{\theta} + 2mr\, \dot{r}\dot{\theta} + mgr\sin\theta = 0 \ ,$$

These are the equations of motion of the system.

Example of calculating a minimum value

Prove that the shortest line between two given points p_1 and p_2 on a cilinder is a helix.
The length S of an arbitrary line on the cilinder between p_1 and p_2 is given by

$$S = \int_{p_1}^{p_2} \left[1 + r^2 \left(\frac{d\theta}{dz} \right)^2 \right]^{1/2} dz \ ,$$

where r, θ and z are the usual cilindrical coordinates for $r = const.$ A relationship between θ and z can be determined for which the last integral has an extremal value by means of

$$\frac{d}{dz} \left(\frac{\partial \phi}{\partial \theta'} \right) - \frac{\partial \phi}{\partial \theta} = 0 \ ,$$

where $\phi = \left[1 + r^2 \theta'^2\right]^{1/2}$ y $\theta' = \frac{d\theta}{dz}$, but since $\partial\phi/\partial\theta = 0$ we have

$$\frac{\partial \phi}{\partial \theta'} = \left(1 + r^2 \theta'^2\right)^{-1/2} r^2 \theta' = c_1 = const. \ ,$$

therefore $r\theta' = c_2$. Thus, $r\theta = c_2 z + c_3$, which is the parametric equation of a helix. Assuming that in p_1 we have $\theta = 0$ and $z = 0$, then $c_3 = 0$. In p_2, make $\theta = \theta_2$ and $z = z_2$, therefore $c_2 = r\theta_2/z_2$, and $r\theta = (r\theta_2/z_2)\, z$ is the final equation.